ON INDEFINITE BINARY QUADRATIC FORMS

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1. If

$$Q(x, y) = a x^2 + b x y + c y^2 = [a, b, c] \tag{1}$$

is an indefinite binary quadratic form with real coefficients in integral variables x, y, not both zero, it is well known that M, the lower bound of $|Q(x, y)|$ (usually called the minimum) satisfies the inequality

$$M \leq D/5^{\frac{1}{2}} \tag{2}$$

where D^2 is the discriminant of Q,

$$D^2 = b^2 - 4ac > 0, \quad D > 0. \tag{3}$$

If equality holds in (2) then Q must be equivalent to a multiple of the form $[1, 1, -1]$.

This is part of a famous theorem due to Markoff [4], a considerably simplified proof of which has lately been given by Cassels [2].

Put briefly, Markoff's theorem states that

$$\overline{\lim}\ M/D = \tfrac{1}{3} \tag{4}$$

if we consider all classes of forms with discriminant D^2, and that any form Q with $3M(Q) > D$ is equivalent to a multiple of one of a denumerable set of forms, the Markoff forms, of which the first is $[1, 1, -1]$, the second $[1, 2, -1]$, the third $[5, 11, -5]$.

Recently Barnes [1] has discussed the problem of obtaining corresponding bounds for the product

$$Q(x, y)\ Q(u, v) \tag{5}$$

over integers x, y, u, v such that