A FINITENESS THEOREM FOR CUSPS

BY

DENNIS SULLIVAN

I.H.E.S., Bures-Sur-Yvette, France

Much is known about the topology and geometry of the quotient of hyperbolic 3-space $\mathbf{H}^3$ by the action of a group Γ of isometries which has a fundamental polyhedron with finite volume or (more generally) with a finite number of faces.(1) On the other hand, if the group Γ is merely assumed to be finitely generated, the 3-dimensional fundamental polyhedron may be of *infinite geometrical complexity* (Greenberg, 1966) and few general results about $\mathbf{H}^3/\Gamma$ are known.

Salient among the known results is the Finiteness theorem of Ahlfors (1964) which describes how the 3-dimensional fundamental polyhedron of a finitely generated Kleinian group intersects the domain of discontinuity at infinity in a *finite* polygon. The intersection of the fundamental polyhedron with the limit set at infinity was shown to have spherical measure zero for the case of a finitely generated Kleinian group in Ahlfors (1965) and for the case of a general finitely generated discrete group in Sullivan (1978).

Conjecturally much more is true about this intersection: *the fundamental polyhedron of a finitely generated group should intersect the limit set in only finitely many inequivalent points*. For this conjecture to be true it is necessary that $\mathbf{H}^3/\Gamma$ have finitely many cusps. The *finiteness of the number of cusps* will be proved in this paper.

By definition a cusp of Γ or of $\mathbf{H}^3/\Gamma$ is a conjugacy class of non-trivial maximal parabolic subgroups. There are two types, rank one cusps and rank two cusps. Each (torsion free) rank one cusp determines a part of the manifold $\mathbf{H}^3/\Gamma$ homeomorphic to a cylinder $\times$ ray. Each (torsion free) rank two cusp determines an end of the manifold $\mathbf{H}^3/\Gamma$ homeomorphic to a torus $\times$ ray. This is due to Margulis and is discussed in Thurston's notes [7], section 5.10.

Our analysis begins with the simple fact that the inverse image in $\mathbf{H}^3$ (thought of as the unit ball B in Euclidean 3-spaces) of the various cusps in $\mathbf{H}^3/\Gamma$ consists of a disjoint collection of horoballs (smaller balls in B tangent to the boundary).

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(1) For a good survey see Marden (1977) and Thurston (1978).