NECESSARY CONDITIONS FOR LOCAL SOLVABILITY OF HOMOGENEOUS LEFT INVARIANT DIFFERENTIAL OPERATORS ON NILPOTENT LIE GROUPS

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1. Introduction and allegro

A differential operator L is *locally solvable* at a point x_0 if there exists a neighborhood U of x_0 such that

$$Lu(x) = f(x), \quad \text{all } x \in U,$$

has a solution $u \in C^\infty(U)$ for any $f \in C_0^\infty(U)$. We shall give necessary conditions for local solvability for some classes of left invariant differential operators on nilpotent Lie groups.

Let G be a connected, simply connected, nilpotent Lie group which admits a family of dilations δ_r, $r>0$, which are automorphisms. The δ_r extend to automorphisms of the complexified universal enveloping algebra $U(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. The elements of $U(\mathfrak{g})$ may be identified with the left invariant differential operators on G. An element $L \in U(\mathfrak{g})$ is homogeneous of degree d if $\delta_r(L) = r^d L$, all $r>0$. We equip G with a norm, $|\ |$, which is homogeneous in the sense that if $U_s = \{x \in G: |x| < s\}$, then $\delta_r(U_s) = U_{rs}$.

We shall prove two main theorems concerning the local solvability of a homogeneous element $L \in U(\mathfrak{g})$, with transpose L^τ. The first says that L is unsolvable if $\ker L^\tau$ contains a function in $\mathcal{S}(G)$, the Schwartz space of G. The second result uses the first to obtain a representation-theoretic criterion for unsolvability of L. Let $\hat{G}$ be the set of all irreducible unitary representations of G. If there is an open subset of representations π in $\hat{G}$ such that

(1) $\ker \pi(L^\tau)$ contains a nonzero C^∞ vector, and

(2) $\ker \pi(L^\tau)$ varies smoothly with π,

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