$B(\mathcal{H})$ DOES NOT HAVE THE APPROXIMATION PROPERTY

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In this paper we prove that $B(\mathcal{H})$, the space of all bounded linear operators on a Hilbert space, does not have the approximation property (abbreviated throughout AP).

The first example of a Banach space which does not have AP, was given by P. Enflo [2]. Following the work of Enflo, several other counterexamples to the AP have been constructed.

$B(\mathcal{H})$ is the first Banach space appearing naturally in analysis which is proved to fail AP. $B(\mathcal{H})$ is also the first known example of a C^*-algebra without AP. Our result implies, of course, the existence of a separable C^*-algebra without AP (cf. Corollary on p. 92). Approximation problems in the context of C^*-algebra theory have been considered by several authors (cf. [1], [4], [5], [8], [9]). Let us mention two of these results:

In [4], U. Haagerup proved that the C^*-algebra generated by the left regular representation of the free group on two generators, does have the AP. For some time this C^*-algebra was a candidate for a "natural counterexample" to the AP.

In [9], S. Wasserman proved that $B(\mathcal{H})$ is not nuclear, thus failing the "completely positive approximation property". The latter property, much stronger than AP, is a C^*-algebra analogue of the AP.

Let us now briefly describe the contents of the present paper. It is divided into 5 sections.

In Section 1 we present a criterion for a Banach space not to have the AP. This criterion is a modified version of Enflo's original one. We show how it is related to the ideas of Grothendieck [3], using the tensor product notation, which was originally used in [3] for the purpose of AP but has been neglected since. It seems to the author that the use of this notation makes an essential simplification in several computations.

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