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The deformation theorem for flat chains

by

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The deformation theorem of Federer and Fleming [FF] is a fundamental tool in geometric measure theory. The theorem gives a way of approximating (in the so-called flat norm) a very general k-dimensional surface (flat chain) A in $\mathbf{R}^N$ by a polyhedral surface P consisting of k-cubes from a cubical lattice in $\mathbf{R}^N$. Unfortunately, the theorem requires the original surface to have finite mass and finite boundary mass. In this paper, we remove these finiteness restrictions. That is, we show (in Theorem 1.1, Corollaries 1.2 and 1.3) that the Federer–Fleming deformation procedure gives good approximations to an arbitrary flat chain A. Also, the approximating polyhedral surface depends only on the way in which typical translates of A intersect the $(N-k)$-skeleton of the dual lattice. This lets us answer several open questions about flat chains:

(1) For an arbitrary coefficient group, a nonzero flat k-chain cannot be supported in a set of k-dimensional measure 0.

(2) For an arbitrary coefficient group, a flat chain of finite mass and finite size must be rectifiable. In particular, for any discrete group, finite mass implies rectifiability.

(3) Let G be a normed group with $\sup\{|g|:g\in G\}=\lambda<\infty$. Then for any flat chain with coefficients in G, $\mathrm{M}(A)\leqslant\lambda\mathcal{H}^k(\operatorname{spt} A)$.

(Special cases of (1) and (2) are mentioned as open questions in [Fl], and the special case $G=\mathbf{Z}_p$ of (3) is mentioned as an open question in [Fe1]. Federer and Fleming [FF] proved (1) for real flat chains (and therefore also for integral flat chains). Almgren [A] introduced the notion of size and proved (2) for real flat chains; Federer [Fe2] then gave a much shorter proof.)

Furthermore, in another paper [W3] we use the deformation theorem proved here to give a simple necessary and sufficient condition on a coefficient group G in order for every finite-mass flat chain to be rectifiable.

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