An example of a weakly hyperbolic Cauchy problem not well posed in C^∞

by

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§ 1. Introduction

In this paper we deal with the Cauchy problem

$$u_{tt}-a(t)\,u_{xx}=0, \quad u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x), \tag{1}$$

where $0\leq t\leq T, x\in \mathbf{R}$ and $a(t)$ is a C^∞ function on the interval $[0,T]$ satisfying the assumption

$$a(t)\geq 0. \tag{2}$$

Our purpose is to show that (1) may be *not well posed* in the class $\mathscr{E}(\mathbf{R}_x)$ of the C^∞ functions, contrary to what occurs when $a(t)\geq\lambda>0$.

More precisely, we shall construct a C^∞ function $a(t)$, strictly positive on $[0,\varrho[$ and identically null on $[\varrho,+\infty[$ where ϱ is a given positive number, and two C^∞ functions $\varphi(x)$ and $\psi(x)$ in such a way that (1) has no solution in the class of distributions on $\mathbf{R}_x\times[0,T[$ as $T>\varrho$.

By virtue of the strict positivity of the coefficient $a(t)$ for $t<\varrho$, this problem has a C^∞ solution on $[0,\varrho[\times\mathbf{R}_x$, which is the *unique* solution in the class $C^1([0,\varrho[,\mathscr{D}'(\mathbf{R}_x))$. However, this solution cannot be continued as distribution on any strip $]0,\varrho+\varepsilon[\times\mathbf{R}_x$, $\forall\varepsilon>0$. In particular it does not belong to $C([0,\varrho],\mathscr{D}'(\mathbf{R}_x))$.

Let us recall that problem (1) is said to be *well posed* in some class $\mathscr{F}(\mathbf{R}_x)$ of *real* functions or distributions (or analytic functionals) if, for any φ and ψ in $\mathscr{F}(\mathbf{R}_x)$, it admits a unique solution u in $C^1([0,T],\mathscr{F}(\mathbf{R}_x))$ and the mapping $(\varphi,\psi)\mapsto u$ is continuous.

The equation $u_{tt}-a(t)\,u_{xx}=0$ is called *hyperbolic* (see Mizohata, [2]) when the corresponding Cauchy problem is well posed in $\mathscr{E}(\mathbf{R}_x)$.