REARRANGEMENTS OF C_1-SUMMABLE SERIES

BY

F. BAGEMIHL and P. ERDÖS

Institute for Advanced Study, Princeton, N.J., and University of Notre Dame, Notre Dame, Ind.

§ 1. In this paper we shall be concerned with infinite series whose terms are real numbers. Suppose that the series

$$(1) \qquad \sum_{n=1}^{\infty} a_n$$

is absolutely convergent and has the sum s. Then, as is well known, every rearrangement, $\sum_{n=1}^{\infty} a_n'$, of (1) also converges and has the same sum s. If, however, (1) converges, but not absolutely, then, according to Riemann's classical rearrangement theorem [3, p. 235, or 2, p. 318], for every real number s', there exists a rearrangement of (1) whose sum is s'.

Assume, now, that (1) is C_1-summable [1, p. 7, or 2, p. 464], and that its C_1-sum is σ. Consider the set of all C_1-summable rearrangements of (1); what is the nature of the corresponding set of C_1-sums? We are going to answer this question; the answer turns out to be somewhat more complicated than Riemann's rearrangement theorem (and also more difficult to obtain). We shall show, namely, that, *for any C_1-summable series* (1), *the rearrangement set* (*cf.* Definition 1 below) *consists either of a single number, or of all numbers of the form $\alpha+\nu\beta$ ($\nu=0, \pm 1, \pm 2, \ldots$) for some particular real numbers $\beta \neq 0$ and α, or of all the real numbers. Moreover, given any α, there exists a C_1-summable series* (1) *whose rearrangement set consists of the single number α; and, given any $\beta \neq 0$ and α, there exists a C_1-summable series* (1) *whose rearrangement set consists of all numbers of the form $\alpha+\nu\beta$ ($\nu=0, \pm 1, \pm 2, \ldots$).*

We introduce

Definition 1. *The set of numbers ϱ such that the C_1-sum of some rearrangement of* (1) *is ϱ, will be denoted by R and called the rearrangement set of* (1).