Acta Math., 188 (2002), 263–302

The Schrödinger operator on the energy space: boundedness and compactness criteria

by

VLADIMIR G. MAZ'YA and IGOR E. VERBITSKY

Linköping University
Linköping, Sweden

University of Missouri
Columbia, MO, U.S.A.

1. Introduction

We characterize the class of measurable functions (or, more generally, real- or complex-valued distributions) V such that the Schrödinger operator $H=-\Delta+V$ maps the energy space $\mathring{L}_2^1(\mathbf{R}^n)$ to its dual $L_2^{-1}(\mathbf{R}^n)$. Similar results are obtained for the inhomogeneous Sobolev space $W_2^1(\mathbf{R}^n)$. In other words, we give a complete solution to the problem of the relative form-boundedness of the potential energy operator V with respect to the Laplacian $-\Delta$, which is fundamental to quantum mechanics. Relative compactness criteria for the corresponding quadratic forms are established as well. We also give analogous boundedness and compactness criteria for Sobolev spaces on domains $\Omega\subset\mathbf{R}^n$ under mild restrictions on $\partial\Omega$.

One of the main goals of the present paper is to give necessary and sufficient conditions for the classical inequality

$$\left|\int_{\mathbf{R}^n}|u(x)|^2\,V(x)\,dx\right|\leqslant \text{const}\int_{\mathbf{R}^n}|\nabla u(x)|^2\,dx,\quad u\in C_0^\infty(\mathbf{R}^n), \tag{1.1}$$

to hold. Here the "indefinite weight" V may change sign, or even be a complex-valued distribution on $\mathbf{R}^n$, $n\geqslant 3$. (In the latter case, the left-hand side of (1.1) is understood as $|\langle Vu,u\rangle|$, where $\langle V\cdot,\cdot\rangle$ is the quadratic form associated with the corresponding multiplication operator V.) We also characterize an analogous inequality for the inhomogeneous Sobolev space $W_2^1(\mathbf{R}^n)$, $n\geqslant 1$:

$$\left|\int_{\mathbf{R}^n}|u(x)|^2\,V(x)\,dx\right|\leqslant \text{const}\int_{\mathbf{R}^n}[|\nabla u(x)|^2+|u(x)|^2]\,dx,\quad u\in C_0^\infty(\mathbf{R}^n). \tag{1.2}$$

The second author was supported in part by NSF Grant DMS-0070623.