The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian

by

L. CAFFARELLI, L. NIRENBERG and J. SPRUCK

University of Chicago
Chicago, IL, U.S.A.

Courant Institute, NYU
New York, NY, U.S.A.

University of Massachusetts
Amherst, MA, U.S.A.

Dedicated to Lars Gårding on his 65th birthday

This is a sequel to [1] and [2]. We will study the Dirichlet problem in a bounded domain Ω in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$:

$$\begin{aligned} F(D^2u) &= \psi \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The function F is of a very special nature. It is represented by a smooth symmetric function $f(\lambda_1, \ldots, \lambda_n)$ of the eigenvalues $\lambda=(\lambda_1, \ldots, \lambda_n)$ of the Hessian matrix $D^2u=\{u_{ij}\}$, which we denote by $\lambda(u_{ij})$. The equation is assumed to be elliptic for the functions under consideration, i.e.

$$\frac{\partial f}{\partial \lambda_i} > 0, \quad \forall i \tag{2}$$

and to satisfy:

$$f \text{ is a concave function.} \tag{3}$$

As we will see in section 3, this means F is a concave function of the arguments $\{u_{ij}\}$.

The function f will be required to satisfy various conditions. First of all it is assumed to be defined in an open convex cone $\Gamma \subsetneqq \mathbf{R}^n$, with vertex at the origin, containing the positive cone: $\{\lambda \in \mathbf{R}^n \mid \text{each component } \lambda_i > 0\}$, and to satisfy (2), (3) in