Partition relations for partially ordered sets

by

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Let P be a partially ordered set. If $r<\omega$, then $[P]^r$ denotes the set of all sequences $\langle a_1, \ldots, a_r\rangle$ such that $a_1, \ldots, a_r \in P$ and $a_1 <_P \ldots <_P a_r$. If γ is an ordinal and if α_i, $i<\gamma$ are order types (isomorphism types of linearly ordered sets), then the symbol $P\to(\alpha_i)^r_{i<\gamma}$ means that for any partition $[P]^r=\bigcup_{i<\gamma}K_i$ there exists an $i<\gamma$ and a chain $A\subseteq P$ such that $\operatorname{tp}A=\alpha_i$ and $[A]^r\subseteq K_i$. The negation of the partition symbol is written with $\nrightarrow$ instead of $\to$. Note that if P is a linearly ordered set, then $[P]^r$ and $P\to(\alpha_i)^r_{i<\gamma}$ have the usual meanings. If $\alpha_i=\alpha$ for all $i<\gamma$, then we write $P\to(\alpha)^r_\gamma$ instead of $P\to(\alpha_i)^r_{i<\gamma}$.

This paper is a study of the partition symbol $P\to(\alpha_i)^r_{i<\gamma}$ for partially ordered sets P such that $P\to(\varkappa)^1_\varkappa$ for some infinite cardinal $\varkappa$. Our main result for the case $\varkappa=\omega$ is the following theorem which proves a conjecture of Galvin [10; p. 718].

THEOREM 1. *Let P be a partially ordered set such that $P\to(\omega)^1_\omega$. Then*

$$P\to(\alpha)^2_k \quad \textit{for all } \alpha<\omega_1 \textit{ and } k<\omega.$$

This theorem completes a rather long list of weaker results: Erdös–Rado [7], [8], Hajnal [11], Galvin [9], Prikry [21], Baumgartner–Hajnal [1] and Galvin [10]. The history of the problem is discussed in [1; pp. 193–194], [4; pp. 271–272] and [10; pp. 711–712]. The most general results previously obtained in the direction of Theorem 1 are a result of Baumgartner and Hajnal [1] who proved Theorem 1 for the case when P is a linearly ordered set, and a result of Galvin [10; p. 714] who proved Theorem 1 under the stronger hypothesis $P\to(\eta)^1_\omega$. The hypothesis $P\to(\omega)^1_\omega$ in Theorem 1 is known to be necessary since $P\to(\omega,\omega+1)^2$ implies $P\to(\omega)^1_\omega$ (see [10; p. 718]).

Let us now consider a generalization of Theorem 1 to higher cardinals $\varkappa$. Unfortu-