CAUCHY'S THEOREM AND ITS CONVERSE

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1. Let C be a simple closed contour which has a central point z_0. By a 'central point' of a simple closed contour, we mean a point within the contour, such that every radius vector drawn from it to the contour lies wholly in the closed domain bounded by the contour and intersects it in only one point.

The existence of a central point z_0 imposes the restriction that the inside of C be a star with respect to z_0. Among such star domains many, including all convex domains, have the required property for all points z_0.

We shall first prove a form of Cauchy's theorem which imposes restrictions, both on the form of the contour and on the derivative of the function. We then remove these restrictions later on.

The point of affix

$$\zeta = z_0 + \lambda (z - z_0),$$

when z lies on C; and $0 < \lambda < 1$, lies on a similar closed contour lying within C and having z_0 as its central point. Call this contour C_λ.

Let us further suppose that

(i) $f(z)$ is a function of z, which has got a definite finite value at every point of the closed domain which consists of all the straight lines drawn from z_0 to the contour C; and of all contours C_λ, $0 \leq \lambda \leq 1$, save possibly at the point z_0;

(ii) $f(z)$ is one-valued and continuous along every contour C_λ, $0 \leq \lambda \leq 1$; and differentiable along every contour C_λ, $0 < \lambda < 1$, at every point of C_λ;

(iii) the maximum-modulus of $f(z)$ on the contour C_λ is bounded, when λ tends to zero and also when λ tends to unity;

(iv) $f(z)$ is continuous along every straight-line joining z_0 to the contour C, at every point of the straight line;

(v) $f(z)$ is differentiable along every straight line, joining z_0 to the contour C, at every point of the straight line, save possibly at one or both of its end points;