C^{∞} CONVEX FUNCTIONS AND MANIFOLDS OF POSITIVE CURVATURE

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Some years ago, Gromoll and Meyer [15] proved that if M is a complete noncompact Riemannian manifold with everywhere positive sectional curvature, then M is diffeomorphic to Euclidean space and the exponential map $\exp_p$: $TM_p \to M$ is for every point $p \in M$ a proper map. During our recent work on noncompact Kähler manifolds [9]–[11], we realized that these and other results on such Riemannian manifolds would follow quite readily from one existence theorem, namely: on a complete noncompact Riemannian manifold of positive curvature there is a C^{∞} strictly convex exhaustion function τ, that is, a C^{∞} function τ: $M \to [0, +\infty)$ which is proper and is such that all the eigenvalues of its second covariant differential are everywhere positive (Theorem 1(a)). The function τ can in fact be chosen to be (uniformly) Lipschitz continuous on all of M. The existence of a *continuous* strictly convex exhaustion function (see § 1 for the definition of strict convexity of continuous functions) was deduced in [12] from results in [3]. Therefore the main weight of the present existence theorem is the possibility of choosing the function to be C^{∞}: in fact, the existence theorem as stated is deduced in this paper from a general theorem that continuous strictly convex functions can be approximated by C^{∞} strictly convex functions on any Riemannian manifold (Theorem 2). The purpose of this paper is thus to establish the existence theorem and to provide a systematic exposition of the consequences which flow from it.

In the terminology of classical analysis, Theorem 2 is a smoothing theorem for strictly convex functions on arbitrary Riemannian manifolds. It should be pointed out that the usual procedure of smoothing in euclidean space by convoluting with a spherically symmetric kernel does not carry over to this general situation. Moreover, the analogue of

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