ON THE CONSISTENCY OF BOREL'S CONJECTURE

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For X a subset of [0, 1], there is a family of properties which X might have, each of which is stronger than X having Lebesgue measure zero, and each of which is trivially satisfied if X is countable. The main properties in this family (apart from the Lusin-type conditions, which are really meant to be stated in conjunction with $2^{\aleph_0}=\aleph_1$ or with Martin's axiom—see section 1) are

X has strong measure zero, i.e., if $\varepsilon_0, \varepsilon_1, ..., \varepsilon_n, ...$ $(n<\omega)$ is a sequence of positive reals then there exists a sequence $I_0, I_1, ..., I_n, ...$ $(n<\omega)$ of intervals with length $I_n \leqslant \varepsilon_n$ and $X \subseteq \bigcup_{n<\omega} I_n$.

X has universal measure zero, i.e., $f[X]$ has Lebesgue measure zero for each homeomorphism f: $[0, 1]\to[0, 1]$, equivalently, for each nonatomic, nonnegative real valued Baire measure μ on [0, 1], $\mu(X)=0$.

If X has strong measure zero then X has universal measure zero. These are strong restrictions to place on X—a nonempty perfect set, which can of course have measure zero, cannot have either of these properties, hence no uncountable analytic set can have either of these properties. The question thus arises as to whether there exist uncountable sets with these properties.

For universal measure zero sets, the answer is yes—Hausdorff ([8]) constructed in ZFC a universal measure zero set of power $\aleph_1$.

For strong measure zero sets the situation is different. The notion of strong measure zero is due to Borel ([2]), who conjectured ([2], page 123) that

all strong measure zero sets are countable.

In fact, though, uncountable strong measure zero sets can be constructed if the continuum hypothesis is assumed—the Lusin set ([11]), a set, constructed with the aid of the CH_2 having countable intersection with each first category set, has strong measure zero ([18]).

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