BOUNDARY BEHAVIOUR AND NORMAL MEROMORPHIC FUNCTIONS

BY

OLLI LEHTO and K. I. VIRTANEN

in Helsinki

Introduction

1. This paper deals with the boundary behaviour of meromorphic functions. The considerations lead in a natural manner to a conformally invariant class of meromorphic functions, distinguished by a number of interesting properties, which we call *normal* meromorphic functions. Their definition reads as follows: If $f(z)$ is meromorphic in a simply connected domain G, then $f(z)$ is normal if and only if the family $\{f(S(z))\}$, where $z' = S(z)$ denotes an arbitrary one-one conformal mapping of G onto itself, is normal in the sense of Montel. In multiply connected domains $f(z)$ is said to be normal if it is normal on the universal covering surface.

Normal meromorphic functions admit the following characterization in terms of the spherical derivative: A non-constant meromorphic $f(z)$ is normal in a domain G, which then necessarily is of hyperbolic type, if and only if there exists a finite constant C so that

$$\frac{|f'(z)|\,|dz|}{1+|f(z)|^2} \leqq C\, d\sigma(z), \tag{1}$$

where $d\sigma(z)$ denotes the element of length in the hyperbolic metric of G.

It follows from the definition that e.g. bounded functions, schlicht functions, and, more generally, functions omitting at least three values, are always normal. On the other hand, all functions of bounded type are not normal.

2. In order to arrive in as natural a way as possible at the concept of a normal function, we devote § 1 to a systematic study of the situation that a meromorphic $f(z)$ possesses an asymptotic value α at a boundary point P but has not the angular limit α at this point. In this case there exists, roughly speaking, a zone containing curves with endpoint at P on which $f(z)$ tends to the limit α. This zone, however, is sharply limited, i.e., there exist