# TO REVERSE A MARKOV PROCESS

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Owing to the symmetry with respect to past and present in the definition of the Markov property, this property is preserved if the direction of time is reversed in a process, but the temporal homogeneity is in general not. Now a reversal preserving the latter is of great interest because many analytic and stochastic properties of a process seem to possess an inner duality and deeper insights into its structure are gained if one can trace the paths backwards as well as forwards, as in human history. Such is for instance the case with Brownian motion where the symmetry of the Green's function and the consequent reversibility plays a leading role. Such is also the case of Markov chains where for instance the basic notion of first entrance has an essential counterpart in last exit, a harder but often more powerful tool. Indeed there are many results in the general theory of Markov processes which would be evident from a reverse point of view but are not easy to apprehend directly.

The question of reversal has of course been considered by many authors.(2) One early line of attack (see e.g., [16]) hinged on finding a stationary distribution for the process; once such a distribution is found it is relatively easy to calculate the transition probabilities of the stationary process reversed in time. A more general approach is to reverse the process $\{X_t\}$ from a random time $\alpha$ to get a process $\tilde{X}_t = X_{\alpha-t}$. Hunt [8] considered such a reversal from last exit times in a discrete parameter Markov chain. Chung [4] observed that this could be done with more dispatch from the life time of a continuous parameter minimal chain. Going to a general state space, Ikeda, Nagasawa and Sato [10] considered reversal from the life time of certain processes. This was extended by Nagasawa [15], who reversed more general types of processes from $L$-times, natural generalizations of last exit times, and later by Kunita and T. Watanabe [11]. An assumption common to

(1) Research supported in part by the Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR contract F 44620-67-C-0049, at Stanford University, Stanford, California.

(2) No previous literature on reversal is used in this paper.