Analytic capacity and differentiability properties of finely harmonic functions

by

ALEXANDER M. DAVIE and BERNT ØKSENDAL

University of Edinburgh, Scotland

Agder Distriktshøgskole, Kristiansand, Norway

1. Introduction

Let f be a finely harmonic function defined on a finely open set V in the complex plane $\mathbf{C}$. In this paper we investigate the problem: To what extent is f differentiable in V?

There are of course several ways of interpreting the question. Debiard and Gaveau [4], [5] have proved the following: Let $K \subset \mathbf{C}$ be compact and let $H(K)$ denote the uniform closure on K of functions harmonic in a neighbourhood of K. Then $H(K)$ coincides with the set of functions continuous on K and finely harmonic on the fine interior K' of K. And if $g \in H(K)$ is the uniform limit of functions g_n harmonic in a neighbourhood of K, then ∇g_n converges in $L^2(m)$ to a limit ∇g, which does not depend on the sequence chosen. Here and later m denotes planar Lebesgue measure. In the other direction they give an example of a compact set K and a point $x_0 \in K'$ such that $|\nabla g_n(x_0)| \to \infty$ as $n \to \infty$.

It was conjectured by T. J. Lyons (private communication) that $\{\nabla g_n(x)\}$ always converges outside a set of zero *logarithmic* capacity. In section 3 we prove that this conjecture fails: For any compact set E with zero *analytic* capacity, there exists a compact set K with $E \subseteq K'$ and functions g_n harmonic in a neighbourhood of K such that $g_n \to 0$ uniformly on K and $|\partial g_n/\partial \bar{z}| \to \infty$ uniformly on E (Theorem 1).

In section 4 we show that parts of the proof of Theorem 1 can be used to prove the following estimate for analytic capacity γ (Theorem 2): If E, F are compact sets and $0<\alpha<1$, then

$$\gamma(E) \leqslant A_\alpha[\gamma(E \setminus F) + C_\alpha(F)^{1/\alpha}],$$

where C_α is the capacity associated to the potential $|z|^{-\alpha}$ and A_α is a constant depending only on α. This result in turn implies that any compact set of Hausdorff dimension less than 1 is γ-negligible, i.e. negligible with respect to approximation by bounded analytic functions (Theorem 3).