Upper semi-continuous set-valued functions

by

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§ 1. Introduction

A map F from a metric space X to the power set of a metric space Y is said to be upper semi-continuous, if the set $\{x: F(x)\cap H\neq\emptyset\}$ is closed in X, whenever, H is a closed set in Y. Our first aim in this paper is to obtain information about the possible structure of such maps. One special case of an upper semi-continuous map is provided by the inverse image function $F=f^{-1}$, when f is a closed continuous map from Y to X, that is, when f is continuous and maps closed sets in Y to closed sets in X. In 1947 Vaĭnšteĭn [15] announced and in 1952 [16] gave the proof that, in this special case, each set $F(x)=f^{-1}(x)$, with x in X, has a compact boundary. In 1948, Choquet [17] considered upper semi-continuous set-valued functions, under the name strongly upper semi-continuous functions. Choquet expressed the opinion that the condition of strong upper upper semi-continuity is very restrictive; a view that we shall amply justify. He gave, without proof, the result that, if F is an upper semi-continuous map of a metric space X to a metric space Y, then, for each x_0 in X, it is possible to choose a compact set K contained in $F(x_0)$ with the property that for each neighbourhood G of K in Y, there is a neighbourhood U of x_0 in X with

$$F(U)\subset G\cup F(x_0).$$

Had Choquet given the proof of his result, it seems sure that the connection between this result and Vaĭnšteĭn's result would have been apparent. As it was, the connection remained undiscovered for many years.

Following up Vaĭnšteĭn's work, Taĭmanov [14] and Lašnev [7], show that, in the special case of the inverse image function F of a closed continuous function f, the set of x, for which $F(x)$ has a non-empty interior, is a sigma-discrete set in X. More recently, in 1977, in a manuscript [18], that has remained unpublished, S. Dolecki rediscovered Choquet's result, in a slightly different form. He gives some applications, writing with S. Rolewicz in [19] and extensions with A. Lechicki in [20].

In this paper we take the theory rather further. Recall that a family of sets in a