An extended Euler-Poincaré theorem

by

ANDERS BJÖRNER[1] and GIL KALAI[1]

Royal Institute of Technology
Stockholm, Sweden

Hebrew University
Jerusalem, Israel

Dedicated to the memory of Alex Zabrodsky

1. Introduction

Suppose that Γ is a finite simplicial complex and $\mathbf{k}$ a field. Let $f=(f_0, f_1, \ldots)$ and $b=(b_0, b_1, \ldots)$ be the *f-vector* and *Betti sequence* of Γ, i.e., $f_i=\operatorname{card}\{F\in\Gamma \mid \dim F=i\}$ and $b_i=\dim_{\mathbf{k}} \tilde{H}_i(\Gamma, \mathbf{k})$, $i\geqslant 0$. The well-known 1899 theorem of Poincaré [P1, P2] (usually called the *Euler-Poincaré formula*) states that

$$\sum_{i\geqslant 0}(-1)^i f_i = \sum_{i\geqslant 0}(-1)^i b_i. \tag{1.1}$$

It was later shown by Mayer [M] that no other linear relation holds between f and b.

In this paper we introduce d (where $d=\dim\Gamma$) non-linear relations which f and b are shown to satisfy. Also, we prove that (1.1) together with these new relations completely characterize the pairs (f, b) of numerical sequences which arise as f-vectors and Betti sequences of finite simplicial complexes. Several related results are discussed concerning such (f, b)-pairs for simplicial complexes, and a characterization is given of those integer sequences which can arise as Betti sequences of simplicial complexes on at most n vertices.

In recent years f-vectors of various classes of simplicial and polyhedral complexes have been intensively studied. We refer the reader to the surveys [Bj, BK, St]. A basic result is the Kruskal-Katona theorem, which characterizes the f-vectors of arbitrary

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