A COUNTEREXAMPLE TO THE APPROXIMATION PROBLEM IN BANACH SPACES

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A Banach space B is said to have the approximation property (a.p. for short) if every compact operator from a Banach space into B can be approximated in the norm topology for operators by finite rank operators. The classical approximation problem is the question whether all Banach spaces have the a.p. In this paper we will give a negative answer to this question by constructing a Banach space which does not have the a.p. A Banach space is said to have the bounded approximation property (b.a.p. for short) if there is a net (S_n) of finite rank operators on B such that $S_n \to I$ in strong operator topology and such that there is a uniform bound on the norms of the S_n:s. It was proved by Grothendieck that the b.a.p. implies the a.p. and that for reflexive Banach spaces the b.a.p. is equivalent to the a.p. (see [1] p. 181 Cor. 2). So what we actually do in this paper is to construct a separable reflexive Banach space which fails to have a property somewhat weaker than the b.a.p.—the exact statement is given by Theorem 1. Since a Banach space with a Schauder basis has the b.a.p.—for such a space the S_n:s can be chosen to be projections—our example also gives a negative solution of the classical basis problem.

The approach we have used in this paper—to put finite-dimensional spaces together in a combinatorial way—is similar to that of Enflo [2] but in the present paper the constructions are made in higher dimensions. Since we will work with symmetry properties of high-dimensional spaces several considerations which were necessary in [2] can be left out in the present paper.

There are several ways of continuing the work on the same lines as in this paper, it has already been shown that Theorem 1 can be improved in several directions. We will discuss some of these extensions at the end of this paper. There are quite a few equivalent formulations of the approximation problem known and also many consequences of any solution of it. For most of these results the reader is referred to [1] and to papers by W. B. Johnson, H. P. Rosenthal and M. Zippin ([3] and [4]).

If T is an operator on a Banach space B and (M) is a subspace of B, put $\|T\|_{(M)} = \sup_{x \in (M)} \|Tx\|/\|x\|$. We have