THE EULER CLASS OF GENERALIZED VECTOR BUNDLES

BY

J. WOLFGANG SMITH

Oregon State University, Corvallis, Ore., U.S.A.(*)

1. Introduction

Let $\xi = (B, X, \pi)$ denote an oriented vector bundle(1) of dimension n, X being its base space, B its total space and $\pi: B \to X$ the projection. The obstruction to nonzero cross-sections $s: X \to B$ is a distinguished element χ in $H^n(X)$, the n-dimensional singular integral cohomology of X, known as the *Euler class* of ξ. We begin by briefly recollecting how χ may be defined. Let K denote the singular simplicial complex of X, K^* the singular simplicial complex of B, and K^0 the subcomplex of K^* whose $(n-1)$-skeleton lies in $\dot{B}$, the nonzero part of B. One now defines an integral cocycle ε on K^0 in the following manner:(2) Let Δ_n denote the standard n-simplex, $\dot{\Delta}_n$ its boundary, and let $\sigma: \Delta_n \to B$ be a singular n-simplex in K^0. Then $\pi \circ \sigma: \Delta_n \to X$ induces a bundle $\xi' = (B', \Delta_n, \pi')$ over Δ_n, and one may conclude(3) from the fact that Δ_n is contractible that ξ' is equivalent to a product bundle. Consequently there exists a second projection $p: B' \to V_n$, where V_n denotes a standard oriented n-dimensional vector space. Moreover, the map $\sigma: X \to B$ induces a cross-section $s: \Delta_n \to B'$, and since σ maps $\dot{\Delta}_n$ to $\dot{B}$, $p \circ s$ maps $\dot{\Delta}_n$ to $\dot{V}_n$, the punctured vector space. Since $\dot{\Delta}_n$ and $\dot{V}_n$ are homotopically equivalent to the oriented $(n-1)$-sphere, the restriction $p \circ s \,|\, \dot{\Delta}_n$ has a well-defined degree.(4) It is easy to verify that this integer does not depend on the choice of p, and consequently the formula

$$\varepsilon(\sigma) = \text{degree}\,(p \circ s \,|\, \dot{\Delta}_n) \tag{1.1}$$

(*) This research was supported in part by the National Science Foundation under NSF G-23722.

(1) For basic facts regarding vector bundles and characteristic classes we refer to J. Milnor [10].

(2) For basic facts regarding singular homology we refer to Eilenberg and Steenrod [7].

(3) Steenrod [12], Theorem 11.6.

(4) Cf. Eilenberg and Steenrod [7], p. 304.