THE SPECTRUM OF DIFFERENCE OPERATORS AND ALGEBRAIC CURVES

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The explicit linearization of the Korteweg–de Vries equation [10, 18] and the Toda lattice equations [10, 12, 22] led to a theory relating periodic second order (differential and difference) operators to hyperelliptic curves with branch points given by the periodic and antiperiodic spectrum of the original operator. As a result the periodic second order operators with a given spectrum form a torus (except for a lower dimensional submanifold) which is the Jacobi variety of the defining curve. Krichever [15, 16, 17], motivated by further examples in the work of Zaharov-Shabat [30], showed how curves with certain properties lead to commuting differential operators reconfirming forgotten work by Burchnell and Chaundy [6]. Inspired by Krichever's ideas, Mumford [24] establishes then a dictionary between commutative rings of (differential and difference) operators and algebraic curves using purely algebraic methods. As an example, the Hill's operator whose spectrum consists of a finite number of non-degenerate bands leads to a finite number of independent differential operators commuting with the original Hill's operator and this commutative ring defines a curve of finite genus. However, the generic Hill's operator has an infinite number of bands and must be analyzed in terms of a hyperelliptic curve of infinite genus; see McKean and Trubowitz [21]. These analytical techniques have not yet been extended to higher order differential operators so that the correspondence between differential operators and curves, generically of infinite genus, is far from being understood. In view of this, it is important to discuss in detail the correspondence between periodic *difference* operators and algebraic curves (of finite genus). In the second order case, the periodic difference operators are good approximations of the periodic differential operators and

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