TWO NEW INTERPOLATION METHODS BASED ON THE DUALITY MAP

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0. Introduction

In his classical paper [11] (1927) Marcel Riesz proved a theorem on linear operators mapping L_p spaces on one measure space onto L_q spaces on another measure space. In the case when the underlying measure spaces are *finite sets* it can be stated as follows. *Let T be a linear operator mapping functions on one finite set onto functions on another finite set (in other words: an $n \times m$ matrix) and denote by M_{pq} the norm of T considered as an operator $T: L_p \to L_q$ where $p, q \in [1, \infty]$, $p \leqslant q$. Then* $\log M_{pq}$ *is a convex function of the pair* $(1/p, 1/q)$. Several years later his student Olof Thorin [14] (compare also [15]) found a very nice proof based on function theory (three line theorem of Doetch). It works in the complex case only but removes the restriction $p \leqslant q$. Accordingly the theorem is now known as the Riesz–Thorin theorem. It has become a standard tool in many branches of analysis and it has been generalized in many directions (see e.g. [17], chap. 12). The current text-books always give Thorin's proof and Riesz's original proof has fallen into oblivion. The purpose of this paper is to reinterpret Riesz's proof in the light of the theory of *interpolation spaces*.

To show how this is done, we shall first sketch Riesz's proof. Putting $M_0 = M_{p_0, q_0}$, $M_1 = M_{p_1, q_1}$ and $M = M_{p, q}$, where $1/p = 1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and analogously for q, it suffices to show that $M \leqslant M_0^{1-\theta} M_1^\theta$ for some $\theta \in (0, 1)$. Riesz does this by choosing $a \in L_p$ and $\beta \in L_{q'}$ $(1/q + 1/q' = 1)$ with unit norms such that $M = \langle Ta, \beta \rangle$ and combines this choice with suitable Hölder inequalities. (Since we are presently dealing with *finite dimensional spaces*, the question of existence of a and β does not cause any difficulty. Elements of dual spaces we usually denote by Greek letters.) The details can be arranged as follows. By Lagrange multipliers, say, we find $Ta = M \operatorname{grad} \|\beta\|_{q'}$ and $T^t\beta = M \operatorname{grad} \|a\|_p$ so that in par-