ON THE DIOPHANTINE EQUATION $1^k+2^k+\ldots+x^k+R(x)=y^z$

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1. Introduction

In J. J. Schäffer [4] the equation

$$1^k+2^k+\ldots+x^k=y^m \tag{1}$$

is studied. Schäffer proves that for fixed $k>0$ and $m>1$ the equation (1) has an infinite number of solutions in positive integers x and y only in the cases

$$\text{(I) } k=1,\ m=2;\quad \text{(II) } k=3,\ m\in\{2,4\};\quad \text{(III) } k=5,\ m=2.$$

He conjectures that all other solutions of (1) have $x=y=1$, apart from $k=m=2$, $x=24$, $y=70$. In [1], the present authors have extended Schäffer's result by proving that for fixed $r, b\in\mathbf{Z}$, $b\neq 0$ and fixed $k\geqslant 2$, $k\notin\{3,5\}$ the equation

$$1^k+2^k+\ldots+x^k+r=by^z \tag{2}$$

has only finitely many solutions in integers $x, y\geqslant 1$ and $z>1$ and all solutions can be effectively determined. In this paper we prove a further generalization.

THEOREM. *Let $R(x)$ be a fixed polynomial with rational integer coefficients. Let $b\neq 0$ and $k\geqslant 2$ be fixed rational integers such that $k\notin\{3,5\}$. Then the equation*

$$1^k+2^k+\ldots+x^k+R(x)=by^z \tag{3}$$

in integers $x, y\geqslant 1$ and $z>1$ has only finitely many solutions.

The proof of our theorem differs from our proof in [1] in quite a few respects. We combine a recent result of Schinzel and Tijdeman [5] with an older, ineffective theorem by W. J. Le Veque [2]. Thus, we can determine an effective upper bound for z, but not