HYPOELLIPTIC SECOND ORDER DIFFERENTIAL EQUATIONS

BY

LARS HÖRMANDER

The Institute for Advanced Study, Princeton, N.J., U.S.A.

1. Introduction

A linear differential operator P with C^∞ coefficients in an open set $\Omega \subset \mathbf{R}^n$ (or a manifold) is called hypoelliptic if for every distribution u in Ω we have

$$\text{sing supp } u = \text{sing supp } Pu,$$

that is, if u must be a C^∞ function in every open set where Pu is a C^∞ function. Necessary and sufficient conditions for P to be hypoelliptic have been known for quite some time when the coefficients are constant (see [3, Chap. IV]). It has also been shown that such equations remain hypoelliptic after a perturbation by a "weaker" operator with variable coefficients (see [3, Chap. VII]). Using pseudo-differential operators one can extend the class of admissible perturbations further; in particular one can obtain in that way many classes of hypoelliptic (differential) equations which are invariant under a change of variables (see [2]). Roughly speaking the sufficient condition for hypoellipticity given in [2] means that the differential equations with constant coefficients obtained by "freezing" the arguments in the coefficients at a point x shall be hypoelliptic and not vary too rapidly with x.

However, the sufficient conditions for hypoellipticity given in [2] are far from being necessary. For example, they are not satisfied by the equation

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} = f, \tag{1.1}$$

for the operator obtained by freezing the coefficients at a point must operate along a two dimensional plane only so it cannot be hypoelliptic. But Kolmogorov [8] constructed already in 1934 an explicit fundamental solution of (1.1) which is a C^∞ function outside the diagonal, and this implies that (1.1) is hypoelliptic.

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