FACETS AND NONFACETS OF CONVEX POLYTOPES

BY

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1. Introduction

Throughout this paper we shall follow, with very few exceptions, the notation and terminology introduced by Professor B. Grünbaum in [5], and the reader is referred to this work for further information on the properties of convex polytopes. By an *equifacetted d-polytope* we mean any d-dimensional convex polytope in Euclidean space whose facets (that is, faces of dimension $d-1$) are all of the same combinatorial type. Many equifacetted polytopes are known, and we mention, by way of example, three classes of polytopes which have been extensively studied: the regular polytopes [3], the simplicial polytopes [5, §§ 4.5 and 9.2] and the cubical polytopes [5, §§ 4.6 and 9.4]. This paper is concerned with problems of the following kind: If P is a given d-dimensional convex polytope, does there exist an equifacetted $(d+1)$-polytope Q whose facets are all combinatorially equivalent to P? If the answer to this question is in the affirmative, then P will be called a *d-facet* or a *facet*, and if the answer is in the negative, then P will be called a *d-nonfacet* or a *nonfacet*.

In the literature only the case $d=2$ has been mentioned, and the problem of characterising the 2-facets and 2-nonfacets is completely straightforward. It is well known (see, for example, [15, p. 149]) that if a three-dimensional convex polytope Q has p_n 2-faces which are n-gons ($n=3, 4, \ldots$) then

$$3p_3+2p_4+p_5 \geqslant 12. \tag{1}$$

It is therefore impossible for all the 2-faces of Q to be n-gons with $n \geqslant 6$. On the other hand, the tetrahedron, cube, and regular dodecahedron are equifacetted 3-polytopes bounded by triangles, quadrilaterals, and pentagons respectively, so we deduce:

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