ON SIMULTANEOUS APPROXIMATIONS OF TWO ALGEBRAIC NUMBERS BY RATIONALS

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1. Introduction

1.1. *Main results.* Throughout this paper, $\|\xi\|$ will denote the distance of the real number ξ from the nearest integer. We shall prove the following results which represent extensions to simultaneous approximations of Roth's famous theorem [5] on rational approximations to an algebraic irrational α.

THEOREM 1. *Let α, β be algebraic and $1, \alpha, \beta$ linearly independent over the field of rationals $\mathbf{Q}$. Then for every $\varepsilon > 0$ there are only finitely many positive integers q with*

$$\|q\alpha\| \cdot \|q\beta\| \cdot q^{1+\varepsilon} < 1. \tag{1}$$

COROLLARY. *Let $\alpha, \beta, \varepsilon$ be as before. There are only finitely many pairs of rationals p_1/q, p_2/q satisfying*

$$\left|\alpha - \frac{p_1}{q}\right| < |q|^{-3/2-\varepsilon}, \qquad \left|\beta - \frac{p_2}{q}\right| < |q|^{-3/2-\varepsilon}. \tag{2}$$

A dual to Theorem 1 is

THEOREM 2. *Let $\alpha, \beta, \varepsilon$ be as in Theorem 1. There are only finitely many pairs of rational integers $q_1 \neq 0$, $q_2 \neq 0$ with*

$$\|q_1\alpha + q_2\beta\| \cdot |q_1 q_2|^{1+\varepsilon} < 1. \tag{3}$$

COROLLARY. *Again let $\alpha, \beta, \varepsilon$ be as in Theorem 1. There are only finitely many triples q_1, q_2, p of rational integers with $q = \max(|q_1|, |q_2|) > 0$ satisfying*

$$|q_1\alpha + q_2\beta + p| < q^{-2-\varepsilon}. \tag{4}$$

1.2. *Approximations by rationals or quadratic irrationals.* Let ω be either rational or a quadratic irrational. There is a polynomial $f(t) = xt^2 + yt + z \not\equiv 0$, unique up to a factor ± 1,