RANDOM WALKS WITH SPHERICAL SYMMETRY

BY

J. F. C. KINGMAN

Pembroke College, Cambridge, England

1. Introduction

Let $\mathbf{X}=(X_1, X_2, \ldots, X_n)$ be a random n-vector with spherical symmetry, that is, a random variable taking values in Euclidean n-space R^n with the property that, if A is any measurable subset of R^n, and A' is obtained from A by rotation about the origin, then

$$P(\mathbf{X}\in A) = P(\mathbf{X}\in A').$$

Then the distribution of $\mathbf{X}$ is determined by that of its length

$$X = |\mathbf{X}| = \left\{\sum_{i=1}^{n} X_i^2\right\}^{\frac{1}{2}},$$

and in particular the characteristic function of $\mathbf{X}$ is given by

$$\Phi(\mathbf{t}) = E(e^{i\mathbf{t}\cdot\mathbf{X}}) = E(e^{itX\cos\theta}), \tag{1}$$

where $t=|\mathbf{t}|$, and θ is the angle between the vectors $\mathbf{t}$ and $\mathbf{X}$. Clearly θ and X are independent, θ having the distribution which a uniformly distributed unit vector makes with a fixed axis.

It is readily shown that, when $n \geqslant 2$, $\lambda = \cos\theta$ has a probability density

$$f_n(\lambda) = \frac{(\frac{1}{2}n-1)!}{\pi^{\frac{1}{2}}(\frac{1}{2}n-\frac{3}{2})!}(1-\lambda^2)^{\frac{1}{2}n-\frac{3}{2}} \qquad (-1 \leqslant \lambda \leqslant 1). \tag{2}$$

Hence, for any complex u,

$$\begin{aligned} E(e^{iu\cos\theta}) &= \frac{(\frac{1}{2}n-1)!}{\pi^{\frac{1}{2}}(\frac{1}{2}n-\frac{3}{2})!}\int_{-1}^{1} e^{iu\lambda}(1-\lambda^2)^{\frac{1}{2}n-\frac{3}{2}}\,d\lambda \\ &= J_{\frac{1}{2}n-1}(u)\,(\tfrac{1}{2}u)^{-\frac{1}{2}n+1}(\tfrac{1}{2}n-1)! \end{aligned}$$

by the Poisson integral ([16], 48) for the Bessel function $J_{\cdot}(\cdot)$.