A CRITICAL TOPOLOGY IN HARMONIC ANALYSIS ON SEMIGROUPS

BY

ARNE BEURLING

Institute for Advanced Study, Princeton N. J.

Introduction

Throughout this paper S shall denote a discrete Abelian semi-group with an irreducible unit, denoted 0, and with a law of cancellation. Spelled out explicitly the two last conditions read:

$$x_1 + x_2 = 0 \Rightarrow x_1 = x_2 = 0, \tag{1}$$

$$x_1 + y = x_2 + y \Rightarrow x_1 = x_2, \tag{2}$$

for elements x_1, x_2, y belonging to S. A semigroup of this kind possesses a natural partial ordering where $x_1 \leqslant x_2$ means that $y \in S$ exists such that $x_1 + y = x_2$. Since y is unique the notation $x_2 - x_1$ stands for an element in S well defined whenever $x_1 \leqslant x_2$.

On S we postulate the existence of a positive function $\omega(x)$, satisfying the following two conditions:

$$y \leqslant 2x \Rightarrow \omega(y) \leqslant 2\,\omega(x), \tag{3}$$

$$\sum e^{-\lambda_0 \omega(x)} \leqslant 1, \tag{4}$$

where λ_0 is a positive constant. In (4) as in all series in the sequel the summation is extended ower $x \in S$ if no other indication is given. The two previous conditions imply that

$$N(x, S) \equiv \sum_{y \leqslant x} 1 \leqslant e^{2\lambda_0 \omega(x)}. \tag{5}$$

The counting function $N(x, S)$ being finite expresses an intrinsic property of S not shared by all semigroups and particularly not by "half planes" of lattice points considered by Helson and Lowdenslager (cf. [3], [4]).