BOUNDED APPROXIMATION BY POLYNOMIALS

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1. Introduction

In this paper we present a complete solution to the following problem: if G is an arbitrary bounded open set in the complex plane, characterize those functions in G that can be obtained as the bounded pointwise limits of polynomials in G. Roughly speaking, the answer is that a function is such a limit if and only if it has a bounded analytic continuation throughout a certain bounded open set G^* that contains G. This set G^* is the inside of the "outer boundary" of G. More precisely, if G is a bounded open set and if H is the unbounded component of the complement of G^- (the closure of G), then G^* denotes the complement of H^-.

A sequence of polynomials $\{p_n\}$ is said to converge boundedly to a function f in an open set G if the polynomials are uniformly bounded in G, and if $p_n(z)$ converges to $f(z)$ at each point $z \in G$. It follows that f is bounded in G. Also, by the Stieltjes-Osgood theorem (see [8], Chapter II, § 7) the convergence is uniform on compact subsets of G and thus f is analytic in G.

MAIN THEOREM. *Let G be a bounded open set in the plane and let f be a bounded analytic function in G. If there is a function F, analytic in G^* and agreeing with f in G, with $|F(z)| \leqslant M$ in G^*, then there is a sequence of polynomials $\{p_n\}$ such that*

(i) $\lim p_n(z) = F(z) \qquad (z \in G^*)$,

(ii) $|p_n(z)| \leqslant M \qquad (z \in G^*; \quad n = 1, 2, \ldots)$.

Conversely, if there is a sequence of polynomials converging to f at each point of G, and uniformly bounded in G, then there is a bounded analytic function F in G^* that agrees with f in G.

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