INEQUALITIES RELATED TO CERTAIN COUPLES OF LOCAL RINGS

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Let Q be a (Noetherian) local ring with maximal ideal $\mathfrak{m}$, and let $\mathfrak{p}$ be a prime ideal in Q such that $\dim \mathfrak{p} + \operatorname{rank} \mathfrak{p} = \dim Q$. Serre showed, using homological means, that if Q is regular, then the local ring $Q_\mathfrak{p}$ is also regular ([8], Theorem 5, p. 186). Under a special assumption Nagata obtained what might be considered a quantitative extension of this result. He proved that if $\mathfrak{p}$ is analytically unramified, then the multiplicity of $\mathfrak{p}$ is not larger than that of $\mathfrak{m}$ ([5], Theorem 10, p. 221). In the present paper it will be shown that under a slightly different special assumption much more can be said. In fact, under that assumption there holds an inequality between certain sum-transforms of the Hilbert functions of $\mathfrak{p}$ and of $\mathfrak{m}$. One seems free to believe that a similar inequality would hold true also in the general case. To prove this it would suffice to prove an analogous statement concerning flat couples of local rings. We shall actually derive a theorem which implies a particular instance of that statement. As a consequence we obtain a generalization and a new proof of Serre's result. Introducing a natural measure of how much a local ring deviates from being regular, we prove that $Q_\mathfrak{p}$ is not more irregular than Q. Our methods of proof are non-homological in the sense that they do not involve any homological resolutions.

We shall now describe our results more closely.(¹)

Let Q be a local ring with maximal ideal $\mathfrak{m}$. For each non-negative integer n, define $H(\mathfrak{m}; n)$ as the length of the Q-module $\mathfrak{m}^n/\mathfrak{m}^{n+1}$. Put

(¹) The necessary facts about local rings can be found in Nagata's book [6], where however the terminology is different in some respects. In particular the concepts which we have called rank and dimension of an ideal and dimension of a ring, are termed height and depth of an ideal and altitude of a ring. Concerning flatness, which is dealt with in the Sections 18 and 19 of the book, cf. e.g. the appendix of [4].