A strongly nonlinear parabolic initial boundary value problem

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1. Introduction

Let Ω be a bounded open set in $\mathbf{R}^N$ and let Q_T be the cylinder $\Omega\times(0, T)$ with some given $T>0$. We shall consider the following parabolic initial boundary value problem

$$
\text{(P)}\qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t}+Au(x,t)+g(x,t,u(x,t))=f(x,t) \quad \text{in} \quad Q_T \\ \qquad\qquad\qquad\qquad\qquad\qquad u(x,t)=0 \quad \text{in} \quad \partial\Omega\times(0,T) \\ \qquad\qquad\qquad\qquad\qquad\qquad u(x,0)=\psi(x) \quad \text{in} \quad \Omega, \end{cases}
$$

where A is an elliptic second order operator of the divergence form

$$
(1)\qquad Au(x,t)=\sum_{|\alpha|\leqq 1}(-1)^{|\alpha|}D^\alpha A_\alpha(x,t,u(x,t),Du(x,t))
$$

for each $t\in[0, T]$ with the coefficients A_α satisfying the classical Leray—Lions conditions, and g is the strongly nonlinear part satisfying essentially only the condition

$$
g(x,t,s)s\geqq-\lambda(x,t) \quad \text{for all} \quad (x,t)\in Q_T,\ s\in\mathbf{R},
$$

where λ is some given function in $L^1(Q_T)$.

It was shown by P. Hess [3] that the corresponding Dirichlet problem for the elliptic equation

$$
Au(x)+g(x,u(x))=f(x) \quad \text{in } \Omega
$$

under similar conditions admits a weak solution. This result was recently generalised by J. R. L. Webb [11] also for higher order operators A by using new approximation results for Sobolev spaces obtained by L. I. Hedberg [2]. In this note we shall show that the problem (P) has a solution. This result is analogous to the elliptic case. The case where A is a higher order operator seems more complicated. Some results into this direction were obtained by H. Brezis and F. E. Browder [1], but stronger hypothesis on g was then needed.