The Weyl calculus with locally temperate metrics and weights

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1. Introduction

The Weyl calculus of operators, defined by

$$a^w(x, D)\,u(x) = (2\pi)^{-n} \iint a(1/2(x+y), \xi) \exp(i\langle x-y, \xi\rangle)\,u(y)\,dy\,d\xi \tag{1.1}$$

was developed with general classes of symbols by Hörmander [7], generalizing the calculus of Beals and Fefferman [1], [2], [3]. Both the Weyl calculus and the Beals—Fefferman calculus require that the symbols are temperate, so they cannot grow faster than a polynomial at infinity. Thus one can't use the calculus to study, for example, the operator $-\Delta+\exp(|x|^2)$ on $\mathbf{R}^n$, where Δ is the Laplacean. In [5], Feigin introduces symbol classes corresponding to the weight $f(x)^2+|\xi|^2$, where $0<c<f(x)$ satisfies

$$|\text{grad}\, f(x)| \leqq Cf(x)^{1+\delta}, \quad \delta < 1.$$

The symbols may therefore grow exponentially in the x variables. The corresponding operators are required to be properly supported, so that the Schwartz kernels are supported where

$$|x-y| \leqq C\big(f(x)+f(y)\big)^{-\gamma}, \quad \delta < \gamma.$$

This condition makes it possible to get a calculus for the operators.

In this paper, we generalize the results of the Weyl calculus to locally temperate symbols, which are temperate in the ξ variables only. In order to do that we introduce a metric in the x variables, to define neighborhoods over which the symbols are temperate. We use cut-off functions χ supported in the corresponding neighborhood of the

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