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COMPUTING THE TOPOLOGICAL DEGREES VIA SEMI-CONCAVE FUNCTIONALS

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Abstract. We construct two retracts in Banach spaces and compute the topological degree for completely continuous operator by means of semi-concave functional. The results extend and complement the previous conclusions.

1. Introduction

Computation for topological degrees plays a very important role in the fixed point theory, see the references [1]–[9], [11]–[17], [19]–[22] listed in this paper and others. In [14] there are the following results.

Theorem 1.1. *Let Ω be a bounded open set in a real Banach space E, $\theta \in \Omega$ and $A\colon \overline{\Omega} \to E$ be completely continuous. Suppose that*

$$\|Ax\| \leq \|x\|, \quad Ax \neq x, \quad \textit{for all } x \in \partial\Omega,$$

then the topological degree $\deg(I - A, \Omega, \theta) = 1$.

Theorem 1.2. *Let Ω be a bounded open set in infinite dimensional real Banach space E, $\theta \notin \partial\Omega$ and $A\colon \overline{\Omega} \to E$ be completely continuous. Suppose that*

$$\|Ax\| \geq \|x\|, \quad Ax \neq x, \quad \textit{for all } x \in \partial\Omega,$$

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then the topological degree $\deg(I - A, \Omega, \theta) = 0$.

Through these computations for topological degrees the fixed point theorem of domain expansion and compression was deduced. By replacing the norm with uniformly continuous convex even functional in [22], the above two theorems about the computation for topological degrees and the fixed point theorem of domain expansion and compression were extended. Some other interesting results of this kind can be found in [1], [5], [6], [8], [11], [13]. What about the concave functional case? It is the purpose to answer the question in this paper. In particular, the computations for topological degrees, together with the previous ones, will make it flexible to use the topological degrees in a wider variety of situations.

In order to compute the topological degrees, we construct two retracts by the semi-concave functional. The retracts formed by the convex functional are showed in [21], [22].

Let E be a real Banach space with the zero element denoted by θ. For the theory and properties of the topological degree in Banach spaces we refer to [9], [12], [14], [20].

$\alpha: E \to R$ is said to be a semi-concave functional on E, if for all $x \in E$, $\lambda \in [0,1]$ and $M > 0$,

$$\alpha(\lambda x + (1-\lambda)Mx) \geq \lambda\alpha(x) + (1-\lambda)\alpha(Mx).$$

α is bounded if its range of bounded set in E is bounded. Throughout this paper we denote the open ball centered at θ with the radius $R > 0$ by $B_R = \{x \in E \mid \|x\| < R\}$ and $[x]$ stands for $x/\|x\|$ for $x \in E \setminus \{\theta\}$. We make the following two assumptions:

(H_1) $\alpha(\theta) = 0$ and $\alpha(x) > 0$ for $x \neq \theta$;

(H_2) $\alpha(x) \to \infty$ as $\|x\| \to \infty$.

A subset $X \subset E$ is called a retract of E if there exists a continuous mapping $r: E \to X$, a retraction, satisfying $r(x) = x$, $x \in X$. By a theorem due to J. Dugundji [10], every nonempty closed convex subset of E is a retract of E.

2. Main results

Lemma 2.1. *Let* $\alpha: E \to [0,\infty)$ *be a bounded continuous semi-concave functional satisfying* (H_1) *and* (H_2), *then* $\|x\| \to 0$ *as* $\alpha(x) \to 0$.

Proof. If the assertion is false, then there exist δ_0 and $\{x_n\} \subset E$ such that $\alpha(x_n) < 1/n$ and $\|x_n\| \geq \delta_0$. Since $\|nx_n\| \geq n\delta_0 \to \infty$ as $n \to \infty$, we have from (H_2) that $\alpha(nx_n) \to \infty$ as $n \to \infty$ and $\alpha(nx_n) \geq 1$ for sufficiently large n.

It follows from the semi-concavity and (H_1) that

$$\alpha(x_n) = \alpha\left(\frac{1}{n} n x_n\right) \geq \frac{1}{n}\alpha(n x_n),$$

thus $\alpha(x_n) \geq 1/n$ for sufficiently large n, which is a contradiction. $\square$

Lemma 2.2. *Let E be an infinite dimensional normed linear space. If $\alpha\colon E \to [0,\infty)$ is a uniformly continuous, even, semi-concave functional satisfying* (H_1) *and* (H_2), *then $D_R = \{x \in E \mid \alpha(x) \geq R\}$ is a retract of E for all $R > 0$.*

Proof. We divide the proof into the following parts:

(i) Since E is an infinite dimensional normed linear space, there exists an unbounded linear functional f on E and the kernel of f, $E_0 \subsetneqq E$ is a dense subspace. Hence $D_R^{(0)} = D_R \cap E_0$ is dense in D_R.

It follows from (H_2) that $D'_R = \{x \in E \mid \alpha(x) \leq R\}$ is bounded and thus there exists $R^* > 0$ such that $D'_R \subset B_{R^*} = \{x \in E \mid \|x\| < R^*\}$. Since $\operatorname{conv} B'_{R^*} = \operatorname{conv}\{x \in E \mid \|x\| \geq R^*\} = E$ and $B'_{R^*} \subset D_R$, we have $\operatorname{conv} D_R = E$ and from Theorem 2 in [18] that $I|_{D_R}$ has a continuous extension

$$F\colon E \to D_R \cup \operatorname{conv} D_R^{(0)} \subset D_R \cup (\operatorname{conv} D_R \cap E_0) = D_R \cup E_0. \tag{2.1}$$

(ii) Since α is uniformly continuous and $\alpha(\theta) = 0$, there exists $\delta > 0$ such that if $\|y\| \leq \delta$,

$$\alpha(y) \leq R/3 \tag{2.2}$$

and for any $x \in E$,

$$|\alpha(x-y) - \alpha(x)| \leq R/3. \tag{2.3}$$

By Lemma 2.1 for $0 < m < \delta$, there exists $0 < R_1 < R/3$ such that

$$\|x\| < m \quad \text{if } \alpha(x) \leq R_1, \tag{2.4}$$

and thus

$$\alpha(x) > R_1 \quad \text{if } \|x\| = m. \tag{2.5}$$

Moreover, by (2.2)

$$\alpha(x) \leq R/3 \quad \text{if } \|x\| = m. \tag{2.6}$$

(iii) Again by the uniform continuity of α and $\alpha(\theta) = 0$, there exists $0 < \delta_1 < \delta$ such that if $\|y\| \leq \delta_1$,

$$\alpha(y) < R_1/8 \tag{2.7}$$

and for any $x \in E$,

$$|\alpha(x-y) - \alpha(x)| \leq R_1/8. \tag{2.8}$$

One can take x_0 satisfying

$$0 < \|x_0\| \le \delta_1 \tag{2.9}$$

and, in particular, $x_0 \notin F(E)$. If otherwise, by (2.1)

$$\{x \in E \mid 0 < \|x\| \le \delta_1\} \subset F(E) \subset D_R \cup E_0. \tag{2.10}$$

Then for all $x \in E \setminus \{\theta\}$, let $y = \delta_1 x/\|x\|$. Thus $\|y\| = \delta_1$ and $y \in F(E)$. It is easy to see from (2.7) and (2.10) that $y \notin D_R$ and $y \in E_0$. Therefore, $f(y) = 0$ and $f(x) = 0$ which implies $f \equiv \theta$, a contradiction.

(iv) Now we show that $\alpha(y(x)) \neq 0$ for $x \in \{x \in E \mid 0 < \alpha(x - x_0) \le R_1/2\}$, where

$$y(x) = \left(1 - \frac{2\alpha(x - x_0)}{R_1}\right)(x_0 + m[x - x_0]) + \frac{2\alpha(x - x_0)}{R_1} x.$$

If it is false, $y(x) = \theta$ and

$$-x_0 = \frac{2\alpha(x - x_0)}{R_1}(x - x_0) + \left(1 - \frac{2\alpha(x - x_0)}{R_1}\right)(m[x - x_0]).$$

By the semi-concavity, we have

$$\alpha(x_0) = \alpha(-x_0) \ge \frac{2\alpha(x - x_0)}{R_1}\alpha(x - x_0) + \left(1 - \frac{2\alpha(x - x_0)}{R_1}\right)\alpha(m[x - x_0]). \tag{2.11}$$

If $0 < \alpha(x - x_0) \le R_1/4$, it follows from (2.5) and (2.11) that

$$\alpha(x_0) \ge \left(1 - \frac{2\alpha(x - x_0)}{R_1}\right)\alpha(m[x - x_0]) \ge \frac{R_1}{2}; \tag{2.12}$$

If $R_1/4 < \alpha(x - x_0) \le R_1/2$, it follows from (2.11) that

$$\alpha(x_0) \ge \frac{2\alpha(x - x_0)}{R_1}\alpha(x - x_0) \ge \frac{R_1}{8}. \tag{2.13}$$

Thus $\alpha(x_0) \ge R_1/8$ which contradicts to (2.9) and (2.7).

(v) $x \notin D_R$ for $x \in \{x \in E \mid 0 < \alpha(x - x_0) \le R_1/2\}$. In fact, from (2.3) and (2.9) we have $\alpha(x) \le R/3 + \alpha(x - x_0) < R$.

(vi) $\alpha(x) \neq 0$ if $\alpha(x - x_0) \ge R_1/2\}$. In fact, $\alpha(x) = 0$ implies that $x = \theta$ and $\alpha(x_0) = \alpha(\theta - x_0) \ge R_1/2$ which contradicts to (2.9) and (2.7).

(vii) We prove $\alpha(y(x)) \le R$ for $x \in \{x \in E \mid 0 < \alpha(x - x_0) \le R_1/2\}$, where $y(x)$ is as in (iv).

Since $\alpha(x) \le \alpha(x - x_0) + R_1/8 < R_1$ by (2.9) and (2.8), it follows from (2.4) that $\|x\| < m < \delta$. Then by (2.3) we have

$$\alpha(y(x)) \le \alpha\left(\left(1 - \frac{2\alpha(x - x_0)}{R_1}\right)(x_0 + m[x - x_0])\right) + \frac{R}{3}. \tag{2.14}$$

Since

$$\left\|\left(1 - \frac{2\alpha(x - x_0)}{R_1}\right)x_0\right\| \le \|x_0\| \le \delta_1 < \delta$$

by (2.9) and

$$\left\|\left(1-\frac{2\alpha(x-x_0)}{R_1}\right)m[x-x_0]\right\| \le m < \delta,$$

it follows from (2.3) and (2.2) that

$$\alpha\left(\left(1-\frac{2\alpha(x-x_0)}{R_1}\right)(x_0+m[x-x_0])\right) \le \frac{R}{3}+\frac{R}{3}=\frac{2R}{3}. \tag{2.15}$$

We have from (2.14) and (2.15) that $\alpha(y(x)) \le R$.

(viii) By (H_2) there exists $M>0$ such that $\alpha(x) > R+1$ if $\|x\| = M$.

(ix) Denote $W_{R_1} = \{x \in E \mid \alpha(x-x_0) \ge R_1/2\} \setminus D_R$ and define

$$G(x) = \frac{\alpha(M[y(x)]) - R}{\alpha(M[y(x)]) - \alpha(y(x))}y(x) + \frac{R-\alpha(y(x))}{\alpha(M[y(x)]) - \alpha(y(x))}M[y(x)],$$

for $0 < \alpha(x-x_0) \le \frac{R_1}{2}$, where $y(x)$ is as in (iv);

$$G(x) = \frac{\alpha(M[x]) - R}{\alpha(M[x]) - \alpha(x)}x + \frac{R-\alpha(x)}{\alpha(M[x]) - \alpha(x)}M[x],$$

for $x \in W_{R_1}$; and

$$G(x) = x, \quad \text{for } x \in D_R.$$

It is easy to see from (iv)–(viii) that $G\colon E \setminus \{x_0\} \to E$ is well defined.

Obviously, if $\alpha(x-x_0) = R_1/2$, $y(x) = x$; and if $\alpha(x) = R$,

$$\frac{\alpha(M[x]) - R}{\alpha(M[x]) - \alpha(x)}x + \frac{R-\alpha(x)}{\alpha(M[x]) - \alpha(x)}M[x] = x.$$

Therefore G is continuous. Now we show $G(E\setminus\{x_0\}) \subset D_R$. In fact, we have from the semi-concavity of α that if $x \in W_{R_1}$,

$$\alpha(G(x)) \ge \frac{\alpha(M[x]) - R}{\alpha(M[x]) - \alpha(x)}\alpha(x) + \frac{R-\alpha(x)}{\alpha(M[x]) - \alpha(x)}\alpha(M[x]) = R$$

and if $0 < \alpha(x-x_0) \le R_1/2$,

$$\begin{aligned}\alpha(G(x)) \ge{}& \frac{\alpha(M[y(x)]) - R}{\alpha(M[y(x)]) - \alpha(y(x))}\alpha(y(x)) \\ &+ \frac{R-\alpha(y(x))}{\alpha(M[y(x)]) - \alpha(y(x))}\alpha(M[y(x)]) = R.\end{aligned}$$

Let $r = GF\colon E \to D_R$. Obviously, r is continuous and $r(x) = x$, for all $x \in D_R$, i.e. D_R is a retract of E. $\square$

Theorem 2.1. *Let Ω be a bounded open set in infinite dimensional Banach space E with $\theta \in \Omega$. Suppose that $A\colon \overline{\Omega} \to E$ is completely continuous and $\alpha\colon E \to [0,\infty)$ is a bounded, even, uniformly continuous semi-concave functional satisfying* (H_1) *and* (H_2). *If*

$$\alpha(Ax) \ge \alpha(x), \quad Ax \ne x, \quad \text{for all } x \in \partial\Omega, \tag{2.16}$$

then the topological degree $\deg(I-A,\Omega,\theta)=0$.

PROOF. First we assert $\inf_{x\in\partial\Omega}\alpha(x)>0$. If otherwise, there exists $\{x_n\}\subset\partial\Omega$ such that $\alpha(x_n)\to 0$, then by Lemma 2.1 we have $\|x_n\|\to 0$ which contradicts $\theta\in\Omega$.

Let $R=\inf_{x\in\partial\Omega}\alpha(x)$ and $D_R=\{x\in E\mid \alpha(x)\geq R\}$. It follows from $\theta\notin D_R$ that

$$d \triangleq \inf_{x\in D_R}\|x\|>0. \tag{2.17}$$

It follows from the boundedness of α that there exists $R_1>0$ such that

$$\alpha(x)<R_1 \quad \text{for } x\in\overline{\Omega} \tag{2.18}$$

and by (H_2) there exists $M_1>0$ such that

$$\alpha(x)>R_1 \quad \text{if } \|x\|>M_1. \tag{2.19}$$

From (2.17) there exists a constant $M>1$ sufficiently large such that $Md>\sup_{x\in\overline{\Omega}}\|x\|$ and $Md>M_1$. Hence

$$(MD_R)\cap\overline{\Omega}=\emptyset, \tag{2.20}$$

where $MD_R=\{Mx\mid x\in D_R\}$. In fact, if there is $x\in D_R$ such that $Mx\in\overline{\Omega}$, thus $\|Mx\|<Md$ and $\|x\|<d$, a contradiction to (2.17).

Let $H(t,x)=(1-t)Ax+tMAx$, for all $(t,x)\in[0,1]\times\overline{\Omega}$. Obviously, $H\colon[0,1]\times\overline{\Omega}\to E$ is completely continuous. Suppose that there exist $x_0\in\partial\Omega$ and $t_0\in[0,1]$ such that $(1-t_0)Ax_0+t_0MAx_0=x_0$. Obviously $t_0\neq 0$ and we have from $\alpha(Ax_0)\geq\alpha(x_0)\geq R$ that $Ax_0\in D_R$. Hence $\|Ax_0\|\geq d$ by (2.17) and $\|MAx_0\|\geq Md>M_1$. Consequently, it follows from (2.18) and (2.19) that

$$\alpha(MAx_0)>\alpha(x_0). \tag{2.21}$$

From the semi-concavity of α, (2.16) and (2.21) we have

$$\begin{aligned}\alpha(Ax_0)&=\alpha((1-t_0)Ax_0+t_0MAx_0)\\&\geq(1-t_0)\alpha(Ax_0)+t_0\alpha(MAx_0)>(1-t_0)\alpha(x_0)+t_0\alpha(x_0)=\alpha(x_0),\end{aligned}$$

a contradiction. Then by the homotopy invariance property of topological degree, we have

$$\deg(I-MA,\Omega,\theta)=\deg(I-A,\Omega,\theta). \tag{2.22}$$

Since D_R is a retract of E by Lemma 2.2, there exists a retraction $r\colon E\to D_R$ satisfying $r(x)=x$, $x\in D_R$. Let $\widetilde{A}=rA$. It is clear that $\widetilde{A}\colon\overline{\Omega}\to D_R$ is completely continuous. From $\alpha(Ax)\geq\alpha(x)\geq R$ for $x\in\partial\Omega$, it follows that $A(\partial\Omega)\subset D_R$. Therefore, $\widetilde{A}x=Ax$ for $x\in\partial\Omega$ and hence

$$\deg(I-M\widetilde{A},\Omega,\theta)=\deg(I-MA,\Omega,\theta). \tag{2.23}$$

If $\deg(I-A,\Omega,\theta) \neq 0$, by (2.22) and (2.23) we have that $\deg(I-M\widetilde{A},\Omega,\theta) \neq 0$ which implies that $M\widetilde{A}$ has a fixed point x^* in Ω. Thus $x^* = M\widetilde{A}x^* \in MD_R$ which contradicts (2.20). $\square$

LEMMA 2.3. *Let $\alpha : E \to [0,\infty)$ be a bounded continuous semi-concave functional satisfying* (H_1) *and* (H_2), *then $D_R = \{x \in E \mid \alpha(x) \le R\}$ is a retract of E for all $R > 0$.*

PROOF. We divide the proof into the following steps:

(i) It follows from (H_2) that D_R is bounded and $D'_R = \{x \in E \mid \alpha(x) \ge R\}$ is nonempty. Taking $R_1 > 0$ such that $D_R \subset \{x \in E \mid \|x\| \le R_1\} \triangleq \overline{B}_{R_1}$ and $D'_R \cap \overline{B}_{R_1} \neq \emptyset$.

Because $\overline{B}_{R_1}$ is a closed convex set, there exists a retraction $g_1 \colon E \to \overline{B}_{R_1}$.

(ii) Since $\alpha(x)$ is bounded, there exists a constant $M > R$ such that $\alpha(x) \le M$ for $x \in D'_R \cap \overline{B}_{R_1}$. From the boundedness of $D_{M+1} = \{x \in E \mid \alpha(x) \le M+1\}$, there is $R_2 > R_1$ such that $\alpha(x) > M+1$ for $x \in \partial B_{R_2}$. Since $\theta \notin D'_R$, define

$$g_2(x) = \frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)}(x - R_2[x]), \quad \text{for all } x \in D'_R \cap \overline{B}_{R_1}.$$

Obviously, g_2 is continuous.

(iii) For $x \in D'_R \cap \overline{B}_{R_1}$ define $g_3(x) = g_2(x) + R_2[x]$ for $\|g_2(x)\| \le R_2$ and $g_3(x) = \theta$ for $\|g_2(x)\| > R_2$.

Now we show that $g_3 \colon D'_R \cap \overline{B}_{R_1} \to E$ is continuous. In fact, if $\|g_2(x)\| \le R_2$, i.e.

$$\begin{aligned} \|g_2(x)\| &= \left\| \frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)}(x - R_2[x]) \right\| \\ &= \frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)}(R_2 - \|x\|) \le R_2, \end{aligned} \tag{2.24}$$

then

$$g_3(x) = \left(\frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)}(\|x\| - R_2) + R_2 \right)[x].$$

These imply $g_3(x) = \theta$ when $\|g_2(x)\| = R_2$ and hence g_3 is continuous.

(iv) Define $g_4(x) = g_3(x)$ for $x \in D'_R \cap \overline{B}_{R_1}$ and $g_4(x) = x$ for $x \in D_R$.

For $x \in \{x \in E \mid \alpha(x) = R\}$, $g_2(x) = x - R_2[x]$ and $\|g_2(x)\| = R_2 - \|x\| < R_2$, and then $g_3(x) = x$. Therefore, $g_4 \colon \overline{B}_{R_1} \to E$ is well defined and continuous.

(v) In the following we prove that $\alpha(g_3(x)) \le R$ for $x \in D'_R \cap \overline{B}_{R_1}$, i.e. $g_4 \colon \overline{B}_{R_1} \to D_R$.

Actually, when $\|g_2(x)\| \ge R_2$, $\alpha(g_3(x)) = 0 \le R$; when $\|g_2(x)\| < R_2$, it follows from $\alpha(x) \ge R$ that

$$\frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)} \ge 1$$

and hence

$$\begin{aligned} g_3(x) &= \frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)}(x - R_2[x]) + R_2[x] \\ &= \bigg(\frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)}\bigg(\frac{\|x\|}{R_2} - 1\bigg) + 1\bigg)R_2[x], \\ x &= \frac{\alpha(R_2[x]) - \alpha(x)}{\alpha(R_2[x]) - R}g_3(x) + \bigg(1 - \frac{\alpha(R_2[x]) - \alpha(x)}{\alpha(R_2[x]) - R}\bigg)R_2[x]. \end{aligned}$$

From the semi-concavity of α and $g_3(x) = MR_2[x]$ where

$$M = \frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)}\bigg(\frac{\|x\|}{R_2} - 1\bigg) + 1 > 0$$

by (2.24), we have

$$\begin{aligned} \alpha(x) &\geq \frac{\alpha(R_2[x]) - \alpha(x)}{\alpha(R_2[x]) - R}\alpha(g_3(x)) + \bigg(1 - \frac{\alpha(R_2[x]) - \alpha(x)}{\alpha(R_2[x]) - R}\bigg)\alpha(R_2[x]), \\ \alpha(g_3(x)) &\leq \frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)}\alpha(x) - \bigg(\frac{\alpha(R_2[x]) - R}{\alpha(R_2[x]) - \alpha(x)} - 1\bigg)\alpha(R_2[x]) = R. \end{aligned}$$

(vi) Let $g(x) = g_4(g_1(x))$, for all $x \in E$, then $g: E \to D_R$ is a retraction. $\square$

Theorem 2.2. *Let* $\alpha : E \to [0, \infty)$ *be a bounded continuous semi-concave functional satisfying* (H$_1$) *and* (H$_2$). *Suppose that* Ω *is a bounded open set in* E *with* $\theta \in \Omega$ *and* $A: \overline{\Omega} \to E$ *is completely continuous. If*

$$\alpha(Ax) \leq \alpha(x), \quad Ax \neq x, \quad \textit{for all } x \in \partial\Omega, \tag{2.25}$$

then the topological degree $\deg(I - A, \Omega, \theta) = 1$.

Proof. The operator A can be extended, yet denoted by A, such that $A: E \to E$ is completely continuous. We divide the proof into the following steps:

(a) Let $R = \sup_{x\in\overline{\Omega}} \alpha(x)$, and hence $R < \infty$ since α is a bounded functional. Obviously,

$$\overline{\Omega} \subset D_R = \{x \in E \mid \alpha(x) \leq R\}. \tag{2.26}$$

By Lemma 2.3, D_R is a retract of E. Take a retraction $g: E \to D_R$ and define the completely continuous operator $A_1 = gA: E \to D_R$. It follows from (2.25) that $\alpha(Ax) \leq \alpha(x) \leq R$, i.e. $Ax \in D_R$ for $x \in \partial\Omega$, therefore, $A_1x = Ax$, for all $x \in \partial\Omega$ and

$$\deg(I - A, \Omega, \theta) = \deg(I - A_1, \Omega, \theta). \tag{2.27}$$

(b) By the boundedness of D_R there exists $R_1 > 0$ such that

$$D_R \subset B_{R_1} = \{x \in E \mid \|x\| < R_1\}. \tag{2.28}$$

For $x \in \{x \in E \mid \|x\| = R_1\}$, we have from $A_1 x \in D_R$ that $\|A_1 x\| < R_1 = \|x\|$, and thus by Theorem 1.1,

$$\deg(I - A_1, B_{R_1}, \theta) = 1. \tag{2.29}$$

(c) Let $\Omega' = B_{R_1} \setminus \overline{\Omega}$, then Ω' is a bounded open set and $\Omega' \neq \emptyset$ by (2.26) and (2.28). Moreover, $\partial\Omega' = \partial B_{R_1} \cup \partial\Omega$.

Let $r = \inf_{x \in \overline{\Omega'}} \|x\|$, then $r > 0$ since $\theta \notin \overline{\Omega'}$. Take $0 < m < 1$ such that $mR_1 < r$. If $x \in \partial\Omega$, then $x \in \overline{\Omega'}$, and hence $\|x\| \geq r$; if $x \in \partial B_{R_1}$, then $\|x\| = R_1 \geq r$. Therefore, we have from $A_1(E) \subset D_R$ and (2.28) that for $x \in \partial\Omega'$, $\|mA_1 x\| < mR_1 < r \leq \|x\|$, i.e. $mA_1 x \neq x$, for all $x \in \partial\Omega'$. Now we will show

$$\deg(I - mA_1, \Omega', \theta) = 0. \tag{2.30}$$

If otherwise, there exists $x_1 \in \Omega'$ such that $mA_1 x_1 = x_1$ which leads to a contradiction, that is, $\|mA_1 x_1\| < mR_1 < r = \inf_{x \in \overline{\Omega'}} \|x\| \leq \|x_1\|$.

(d) Consider the completely continuous homotopy

$$H(t, x) = (1 - t)A_1 x + tmA_1 x, \quad (t, x) \in [0, 1] \times \overline{\Omega'}$$

and suppose that there exist $t_0 \in [0, 1]$ and $x_0 \in \partial\Omega'$ such that $x_0 = H(t_0, x_0)$.

If $x_0 \in \partial B_{R_1}$, we have form $A_1 x_0 \in D_R$ and (2.28) that $\|A_1 x_0\| < R_1$ and $\|mA_1 x_0\| < R_1$, then $\|x_0\| \leq (1 - t_0)\|A_1 x_0\| + t_0\|mA_1 x_0\| < R_1$, a contradiction.

If $x_0 \in \partial\Omega$, it is easy to see that $t_0 \neq 0, 1$. Let $t_1 = (1 - t_0 + mt_0)^{-1}$, then $t_1 > 1$ and $A_1 x_0 = t_1 x_0$. By (H_2) there is $t_2 > t_1$ such that $x_2 = t_2 x_0$ satisfying $\alpha(x_2) > R$. Denote $t' = (t_2 - t_1)/(t_2 - 1)$, thus $0 < t' < 1$ and $A_1 x_0 = t' x_0 + (1 - t')x_2$. By (2.25) and the semi-concavity of α we have

$$\alpha(x_0) \geq \alpha(A_1 x_0) \geq t'\alpha(x_0) + (1 - t')\alpha(x_2),$$

so $\alpha(x_0) \geq \alpha(x_2) > R$ which contradicts (2.26).

By the homotopy invariance property of the topological degree, we have

$$\deg(I - A_1, \Omega', \theta) = \deg(I - mA_1, \Omega', \theta). \tag{2.31}$$

(e) Since $B_{R_1} \setminus (\Omega' \cup \Omega) = \partial\Omega$, A_1 has no fixed points in $B_{R_1} \setminus (\Omega' \cup \Omega)$. Therefore,

$$\deg(I - A_1, B_{R_1}, \theta) = \deg(I - A_1, \Omega', \theta) + \deg(I - A_1, \Omega, \theta). \tag{2.32}$$

Finally, by (2.27), (2.29)–(2.32) we conclude $\deg(I - A, \Omega, \theta) = 1$. $\square$

3. An example

In this section a semi-concave functional is given that satisfies the conditions. Let Banach space E be $C^1[a,b]$ with the norm

$$\|x\| = \max_{a\le t\le b} |x(t)| + \max_{a\le t\le b} |x'(t)| \triangleq \|x\|_C + \|x'\|_C$$

for $x \in C^1[a,b]$ and define $\alpha\colon E \to [0,\infty)$ as follows:

$$\alpha(x) = \|x\|_C + \|x'\|_C^\nu, \quad 0<\nu<1,\ x\in E.$$

(a) α is a semi-concave functional. For all $x \in E$, $\lambda\in[0,1]$ and $M>0$,

$$\alpha(\lambda x + (1-\lambda)Mx) = (\lambda + (1-\lambda)M)\|x\|_C + (\lambda+(1-\lambda)M)^\nu \|x'\|_C^\nu.$$

Since $f(t) = t^\nu (0<\nu<1)$ is concave on $[0,\infty)$,

$$(\lambda+(1-\lambda)M)^\nu = f(\lambda+(1-\lambda)M) \ge \lambda + (1-\lambda)M^\nu,$$

and thus

$$\begin{aligned}&\alpha(\lambda x + (1-\lambda)Mx) \ge (\lambda+(1-\lambda)M)\|x\|_C + (\lambda+(1-\lambda)M^\nu)\|x'\|_C^\nu \\ &= \lambda(\|x\|_C + \|x'\|_C^\nu) + (1-\lambda)(\|Mx\|_C + \|Mx'\|_C^\nu) = \lambda\alpha(x) + (1-\lambda)\alpha(Mx).\end{aligned}$$

(b) α is uniformly continuous and bounded since $\alpha(x) \le \|x\| + \|x\|^\nu$ for $x\in E$. Obviously, $\alpha(-x) = \alpha(x)$, $\alpha(\theta)=0$ and $\alpha(x)>0$ for $x\ne\theta$.

(c) For any $K>1$, if $\|x\| > 2K^{1/\nu}$, then either $\|x\|_C > K^{1/\nu}$ or $\|x'\|_C > K^{1/\nu}$. Therefore, $\alpha(x) \ge \max\{\|x\|_C, \|x'\|_C^\nu\} > K$, i.e. $\alpha(x)\to\infty$ as $\|x\|\to\infty$.

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