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ON LOCAL CHARACTERIZATION OF THE STRONG ŚWIĄTKOWSKI PROPERTY FOR A FUNCTION $f : [a, b] \to \mathbb{R}$

Abstract

In this paper we introduce the idea of a function having the strong Świątkowski property at a point. The main result of this work is proving that the class of functions which have the strong Świątkowski property at each point, is equal to the class of all strong Świątkowski functions.

In 1977 T. Mańk and T. Świątkowski introduced a new class of real functions ([9]). Later this class was called the family of Świątkowski functions. A. Maliszewski has modified the definition ([7]) and introduced the notion of the strong Świątkowski property. He noticed that functions having this property are Darboux and quasi-continuous functions. He also gave an example of a Darboux and quasi-continuous function which doesn't have the strong Świątkowski property. A. Maliszewski showed also many other interesting facts connected with the introduced class. (For example he considered problems connected with sums and products of functions of this class.)

In our paper we shall introduce the "locally" strong Świątkowski property. From the private correspondence we know that this problem was investigated by S. Kowalczyk. (His results relate to the functions such that their domain is an open interval and in the version of S. Kowalczyk the domain of considered functions can not be replaced by the closed interval.)

We apply the classical symbols and notions. By $\mathbb{R}$ ($\mathbb{N}$, $\mathbb{Q}$) we denote the set of real numbers (natural numbers, rational numbers). Moreover let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

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The symbol (a, b) $([a, b], [a, b), (a, b])$ denotes an open (closed, right-side open, left-side open) interval in the case if $a < b$ or $b < a$. If (a, b) or $[a, b]$ is the domain of a function, we suppose that $a < b$.

The symbol λ stands for the Lebesgue measure on the real line.

The symbol $\star$ denotes an equivalence relation in interval $\mathbf{I}$ defined in the following way:

$$x \star y \Longleftrightarrow x - y \in \mathbb{Q}.$$

Then the equivalence class of this relation determined by an element $\alpha \in \mathbf{I}$ is denoted by $[\alpha]_\star$.

The symbol $\operatorname{card}(A)$ stands for the cardinality of the set A. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$.

We take the following definition of a Darboux function ([12], [4], [11]).

A function $F : X \to Y$ (X, Y - topological spaces) is called a *Darboux function* if $F(C)$ is a connected set for each connected set $C \subset X$.

Let $f : X \to \mathbb{R}$ (X is a connected subset of $\mathbb{R}$). We say that a point x_0, which is a right-hand accumulation point of the set X, is a *right-hand Darboux point* of the function f, if for each $\alpha \in \mathrm{L}^+(f, x_0) \setminus \{f(x_0)\}$, $y \in (f(x_0), \alpha)$ and $\delta > 0$ there exist $\xi \in [x_0, x_0 + \delta)$ such that $f(\xi) = y$. In an analogous way we define a left-hand Darboux point of the function f.

We shall say that x_0 is a *Darboux point* of function f if it is simultaneously the right-hand and the left-hand Darboux point of f [1].

A function $f : [a, b] \to \mathbb{R}$ is a *quasi-continuous at a point* $x_0 \in [a, b]$ if for each $\varepsilon > 0$ and $\delta > 0$ there exists a nonempty open set $U \subset (x_0 - \delta, x_0 + \delta)$ such that $f(U) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. If the function f is quasi-continuous at each point, we say that f is quasi-continuous.

By $\mathcal{D}bx$ ($\mathcal{Q}_c$) we shall denote the set of all Darboux (quasi-continuous) functions $f : [a, b] \to \mathbb{R}$.

If $z_n \to z$ and $z_n < z_{n+1} < z$ for all $n \in \mathbb{N}$, we will write $z_n \nearrow z$ ($z \nwarrow z_n$). Similarly, if $z_n \to z$ and $z_n > z_{n+1} > z$ for all $n \in \mathbb{N}$, we will write $z_n \searrow z$ ($z \swarrow z_n$).

Let $f : [a, b] \to \mathbb{R}$. The symbols $\mathrm{L}^-(f, x)$, $\mathrm{L}^+(f, x)$, $\mathrm{L}(f, x)$ denote: the cluster set from the left, the cluster set from the right and the cluster set of the function f at the point $x \in [a, b]$.

Let f be a function. We will denote:

by $\mathcal{C}(f)$ ($\mathcal{C}^+(f)$, $\mathcal{C}^-(f)$) the set of all points of bilaterally (right-hand, left-hand) continuity of the function f;

by $\mathcal{D}(f)$ ($\mathcal{D}^+(f)$, $\mathcal{D}^-(f)$) the set of all points of bilaterally (right-hand, left-hand) discontinuity of the function f;

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[1] This definition agrees with the one given in [5].

by $\mathcal{D}bx(f)$ $(\mathcal{D}bx^+(f), \mathcal{D}bx^-(f))$ the set of all points of bilaterally (right-hand, left-hand) Darboux points of the function f;

by $\mathcal{Q}_c(f)$ the set of all points of quasi-continuity of the function f.

Let $\Gamma(f)$ denote the graph of the function f. If A is a subset of the domain of f, the symbol $f|A$ denotes the restriction of f to A.

A function $f : [a, b] \to \mathbb{R}$ is a *strong Świątkowski function* ([7]) if for any two elements $x, y \in [a, b]$ such that $f(x) \neq f(y)$ and for arbitrary $\alpha \in (f(x), f(y))$ there exists a point $z \in (x, y) \cap \mathcal{C}(f)$ such that $f(z) = \alpha$.

By $\mathcal{S}$ we denote the set of all functions $f : [a, b] \to \mathbb{R}$ having the strong Świątkowski property.

The definitions, lemmas and theorems of this paper we will formulate in the case when the domain of considered functions is a closed interval. Remark that all these results are true also in the case if the domain of functions is an open or one-side open interval.

Definition 1. Let $f : [a, b] \to \mathbb{R}$. We say that a point $x_0 \in (a, b)$ cuts a function f, if there exists a number $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and

$$f((x_0 - \delta, x_0) \cap \mathcal{C}(f)) \subset (-\infty, f(x_0)) \text{ and } f((x_0, x_0 + \delta) \cap \mathcal{C}(f)) \subset (f(x_0), +\infty)$$

or

$$f((x_0 - \delta, x_0) \cap \mathcal{C}(f)) \subset (f(x_0), +\infty)) \text{ and } f((x_0, x_0 + \delta) \cap \mathcal{C}(f)) \subset (-\infty, f(x_0)).$$

Definition 2. Let $f : [a, b] \to \mathbb{R}$. We say that a function f has a right-hand strong Świątkowski property at a point $x_0 \in [a, b)$ (briefly $x_0 \in \mathcal{S}^+(f)$) if: $x_0 \in \mathcal{C}^+(f)$ or the following conditions are satisfied:

1° [2] for any $\alpha \in \mathrm{L}^+(f, x_0) \setminus \{f(x_0)\}$ and $\beta \in (\alpha, f(x_0))$ and arbitrary positive number δ such that $(x_0, x_0 + \delta) \subset (a, b)$

$$f^{-1}(\beta) \cap (x_0, x_0 + \delta) \cap \mathcal{C}(f) \neq \emptyset.$$

2° for any $\alpha \in \mathbb{R}$:
if there exist sequences $\{x_n\}, \{y_n\} \subset (a, b)$ such that

$x_n \searrow x_0 \swarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$,

there exists a sequence $\{z_n\} \subset (a, b)$ such that $\{z_n\} \subset \mathcal{C}(f)$, $z_n \searrow x_0$ and $f(z_n) = \alpha$ for any $n \in \mathbb{N}$.

[2]For the purpose of this paper's discussion it is sufficient to assume at this point that $\mathrm{L}^+(f, x_0)$ is an interval and $f(x_0) \in \mathrm{L}^+(f, x_0)$

In an analogous way we define a left-hand strong Świątkowski property at a point x_0.

Definition 3. Let $f : [a, b] \to \mathbb{R}$. We say that a function f has a left-hand strong Świątkowski property at a point $x_0 \in (a, b]$ (briefly $x_0 \in \mathcal{S}^-(f)$) if: $x_0 \in \mathcal{C}^-(f)$ or the following conditions are satisfied:
1° [3] for any $\alpha \in \mathrm{L}^-(f, x_0) \setminus \{f(x_0)\}$ and $\beta \in (\alpha, f(x_0))$ and arbitrary positive number δ such that $(x_0 - \delta, x_0) \subset (a, b)$

$$f^{-1}(\beta) \cap (x_0 - \delta, x_0) \cap \mathcal{C}(f) \neq \emptyset.$$

2° for any $\alpha \in \mathbb{R}$:
if there exist sequences $\{x_n\}, \{y_n\} \subset (a, b)$ such that
$x_n \nearrow x_0 \nwarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$,
there exists a sequence $\{z_n\} \subset (a, b)$ such that $\{z_n\} \subset \mathcal{C}(f)$, $z_n \nearrow x_0$ and $f(z_n) = \alpha$ for any $n \in \mathbb{N}$.

Definition 4. We say that a function $f : [a, b] \to \mathbb{R}$ has the strong Świątkowski property at a point $x_0 \in [a, b]$ (briefly $x_0 \in \mathcal{S}(f)$):

- $x_0 \in \mathcal{S}^+(f)$, for $x_0 \in [a, b)$;
- $x_0 \in \mathcal{S}^-(f)$, for $x_0 \in (a, b]$;

and moreover for $x_0 \in (a, b)$ we have if x_0 cuts a function f, then $x_0 \in \mathcal{C}(f)$.

Remark 5. Let $f : [a, b] \to \mathbb{R}$. If $x_0 \in [a, b]$ is a point of continuity of f. Then f has a strong Świątkowski property at a point x_0.

Remark 6. Let $f : [a, b] \to \mathbb{R}$. The inclusion $[p, q] \subset \mathcal{S}(f)$ (where $p, q \in [a, b]$ and $p < q$) means that the function f has the strong Świątkowski property at each point $x \in (p, q)$ and has the right-hand (left-hand) strong Świątkowski property at the point p (q).

Analogously we shall understand inclusions $(a, b] \subset \mathcal{S}(f)$, $[a, b) \subset \mathcal{S}(f)$ and $(a, b) \subset \mathcal{S}(f)$.

In ([8], p. 10) A. Maliszewski noticed that it can be easily verified that strong Świątkowski functions are Darboux and quasi-continuous functions. The next lemma shows that an analogous fact holds for these properties considered "locally". Moreover, it is sufficient to assume only condition 1° of above definitions.

[3]For the purpose of this paper's discussion it is sufficient to assume at this point that $\mathrm{L}^-(f, x_0)$ is an interval and $f(x_0) \in \mathrm{L}^-(f, x_0)$

Lemma 7. *Let $f : [a,b] \to \mathbb{R}$. If f satisfies condition 1° of Definition 2 (Definition 3) at a point $x_0 \in [a,b)$ ($x_0 \in (a,b]$), then $x_0 \in \mathcal{D}bx^+(f) \cap \mathcal{Q}_c(f)$ ($x_0 \in \mathcal{D}bx^-(f) \cap \mathcal{Q}_c(f)$).*

Proof. Consider the "right-hand condition". (The other case can be proved analogously.) Let $f : [a,b] \to \mathbb{R}$ and let $x_0 \in [a,b)$. If $x_0 \in \mathcal{C}^+(f)$, then obviously $x_0 \in \mathcal{D}bx^+(f) \cap \mathcal{Q}_c(f)$. So, let us assume that $x_0 \notin \mathcal{C}^+(f)$. Then there exists $\alpha \in \mathrm{L}^+(f,x_0) \setminus \{f(x_0)\}$. Let $\beta \in (\alpha, f(x_0))$ and $\delta > 0$ be an arbitrary real number such that $(x_0, x_0+\delta) \subset (a,b)$. From condition 1° of the Definition 2 we have

$$f^{-1}(\beta) \cap (x_0, x_0+\delta) \cap \mathcal{C}(f) \neq \emptyset.$$

Then, according to J. Lipiński's definition ([5]), $x_0 \in \mathcal{D}bx^+(f)$.

Now, we shall show that $x_0 \in \mathcal{Q}_c(f)$. Let $\varepsilon > 0$ and $\delta_1 > 0$. Moreover, let $\beta^* \in (\alpha, f(x_0)) \cap (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$. According to condition 1° of Definition 2, there exists a point $z \in (x_0, x_0+\delta_1) \cap \mathcal{C}(f)$ such that $f(z) = \beta^* \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$. Consequently, there exists a number $\delta^* > 0$ such that $(z-\delta^*, z+\delta^*) \subset (x_0, x_0+\delta_1)$ and $f((z-\delta^*, z+\delta^*)) \subset (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$. Therefore $x_0 \in \mathcal{Q}_c(f)$. □

According to Lemma 7 we have the following corollaries.

Corollary 8. *Let $f : [a,b] \to \mathbb{R}$. Then $\mathcal{C}(f) \subset \mathcal{S}(f) \subset (\mathcal{D}bx(f) \cap \mathcal{Q}_c(f))$.*

Corollary 9. *If $f : [a,b] \to \mathbb{R}$ is a function such that $[a,b] \subset \mathcal{S}(f)$, then f is Darboux and a quasi-continuous function.*

From [8], [10], [1] and previous definitions we have the following lemma.

Lemma 10. *(a) Let $f : [a,b] \to \mathbb{R}$. If f has the strong Świątkowski property, then the set of all continuity points of f is dense in $[a,b]$.*
(b)[4] Let $f : [a,b] \to \mathbb{R}$. If $[a,b] \subset \mathcal{S}(f)$, then the set of all continuity points of f is dense in $[a,b]$.

By Theorem III.4.3 ([6]) and by Lemma 10 (a) we immediately obtain the following assertion.

Theorem 11. *Let $f : [a,b] \to \mathbb{R}$. If f has the strong Świątkowski property, then $\mathcal{D}(f)$ is a first category set.*

The following theorem is the starting point for some considerations.

[4] This fact was observed by M. Marciniak.

Theorem 12. *A function $f : [a,b] \to \mathbb{R}$ is a strong Świątkowski function if and only if $[a,b] \subset \mathcal{S}(f)$.*

PROOF. **Necessity.** Let $f : [a,b] \to \mathbb{R}$ be a strong Świątkowski function and let $x_0 \in [a,b]$. We shall show that if $x_0 \in [a,b)$, then $x_0 \in \mathcal{S}^+(f)$. If $x_0 \in \mathcal{C}^+(f)$, then obviously $x_0 \in \mathcal{S}^+(f)$. So, we can assume that $x_0 \notin \mathcal{C}^+(f)$.

1° There exists $\alpha_1 \in \mathrm{L}^+(f,x_0) \setminus \{f(x_0)\}$. Suppose, for instance, that $\alpha_1 > f(x_0)$. Let $\beta \in (f(x_0),\alpha_1)$ and $\delta > 0$. There exists a sequence $\{x_n\}$ such that $x_n \searrow x_0$ and $f(x_n) \to \alpha_1$. Without loss of generality, we may assume that $(x_0, x_0+\delta) \subset (a,b)$. Then there exists $n_0 \in \mathbb{N}$ such that $x_n \in (x_0, x_0+\delta)$, for each $n \geq n_0$. Moreover, there exists $m_0 > n_0$ such that $f(x_{m_0}) > \beta$. So $f(x_0) < \beta < f(x_{m_0})$. Since $f \in \mathcal{S}$, there exists $x_1 \in (x_0, x_{m_0}) \cap \mathcal{C}(f) \subset (a,b) \cap \mathcal{C}(f)$ such that $f(x_1) = \beta$. Thus we have

$$f^{-1}(\beta) \cap (x_0, x_0+\delta) \cap \mathcal{C}(f) \neq \emptyset.$$

2° Let $\alpha \in \mathbb{R}$ and $\{q_n\}$, $\{y_n\}$ be sequences such that $q_n \in (a,b)$, $y_n \in (a,b)$, for each $n \in \mathbb{N}$ and $q_n \searrow x_0 \swarrow y_n$, $f(q_n) \searrow \alpha \nwarrow f(y_n)$. Let $q_{m_1} > x_0$ be an arbitrary element of the sequence $\{q_n\}$. Choose an element $y_{s_1} \in (x_0, q_{m_1})$. Then $f(y_{s_1}) < \alpha < f(q_{m_1})$. The fact that f has the strong Świątkowski property guarantees existence of $z_1 \in (y_{s_1}, q_{m_1}) \cap \mathcal{C}(f)$ such that $f(z_1) = \alpha$.

Suppose that we have chosen q_{m_k}, y_{s_k}, z_k (for $k = 1,2,...,n-1$) such that $m_k > m_{k-1}$, $s_k > s_{k-1}$ for $k = 2,3,...,n-1$ and $q_{m_k} \in (x_0, y_{s_{k-1}})$, $y_{s_k} \in (x_0, q_{m_k})$, $z_k \in (y_{s_k}, q_{m_k}) \cap \mathcal{C}(f)$ and $f(z_k) = \alpha$ for $k = 2,3,...,n-1$. Now let $q_{m_n} \in (x_0, y_{s_{n-1}})$ and $y_{s_n} \in (x_0, q_{m_n})$. Then $f(y_{s_n}) < \alpha < f(q_{m_n})$. There exists $z_n \in (y_{s_n}, q_{m_n}) \cap \mathcal{C}(f)$ such that $f(z_n) = \alpha$. In this way we define an infinite sequence $\{z_n\}$ such that $\{z_n\} \subset \mathcal{C}(f)$ and $y_{s_n} < z_n < q_{m_n}$, for each $n \in \mathbb{N}$. Since $q_{m_n} \searrow x_0 \swarrow y_{s_n}$. Therefore $z_n \searrow x_0$. Moreover, $f(z_n) = \alpha$ for each $n \in \mathbb{N}$.

In the analogous way we can show: if $x_0 \in (a,b]$, then $x_0 \in \mathcal{S}^-(f)$. To finish the proof of necessity we shall show that if $x_0 \in (a,b)$ cuts a function f, then x_0 is a continuity point of f. Suppose, that x_0 cuts a function f and let $\delta^* > 0$ be a real number such that $(x_0-\delta^*, x_0+\delta^*) \subset (a,b)$ and let, for instance,

$$f((x_0-\delta^*, x_0) \cap \mathcal{C}(f)) \subset (-\infty, f(x_0)) \text{ and } f((x_0, x_0+\delta^*) \cap \mathcal{C}(f)) \subset (f(x_0), +\infty).$$

Let $x_1 \in (x_0-\delta^*, x_0) \cap \mathcal{C}(f)$ and $x_2 \in (x_0, x_0+\delta^*) \cap \mathcal{C}(f)$ (see Lemma 10). Therefore $f(x_1) < f(x_0) < f(x_2)$. Since $f \in \mathcal{S}$, there exists $\hat{x} \in (x_1,x_2) \cap \mathcal{C}(f)$ such that $f(\hat{x}) = f(x_0)$. From the inclusions $f((x_1,x_0) \cap \mathcal{C}(f)) \subset (-\infty, f(x_0))$ and $f((x_0,x_2) \cap \mathcal{C}(f)) \subset (f(x_0), +\infty)$, we can infer that $f(x) \neq f(x_0)$, for any $x \in ((x_1,x_2) \setminus \{x_0\}) \cap \mathcal{C}(f)$. Hence $\hat{x} = x_0$. So $x_0 \in \mathcal{C}(f)$.

Sufficiency. It is sufficient to show that

$$\begin{aligned}&\text{if } f \in \mathcal{D}bx \text{ and } [a,b] \subset \mathcal{S}(f) \text{ and no discontinuity point of } f\\ &\text{cuts the function } f, \text{ then } f \in \mathcal{S}.\end{aligned} \tag{1}$$

(Note that, according to the Corollary 8 and Definition 4, in the case of our Theorem the assumptions of the above implication are fulfilled.) Let $x^*, y^* \in [a,b]$ be real numbers such that $x^* < y^*$ and $f(x^*) \neq f(y^*)$. Let $\eta \in (f(x^*), f(y^*))$. Assume, for instance, that $f(x^*) < \eta < f(y^*)$. We shall show that

$$\text{there exists } u \in (x^*, y^*) \cap \mathcal{C}(f) \text{ such that } f(u) = \eta. \tag{2}$$

Suppose, to the contrary, that condition (2) doesn't hold. Let us define

$$x_0 = \inf\{x \in [x^*, y^*] : f(x) > \eta\}. \tag{3}$$

First we shall show that

$$f(x_0) \leq \eta. \tag{4}$$

In the opposite case $x_0 > x^*$ and we have

$$f([x^*, x_0]) = f([x^*, x_0)) \cup \{f(x_0)\} \subset (-\infty, \eta) \cup \{f(x_0)\}.$$

This means that $f([x^*, x_0])$ is not a connected set, which is impossible because f possesses Darboux property. The proof of (4) is finished.

Moreover, we can observe that $x_0 < y^*$. According to the definition of x_0 we have

$$\begin{aligned}&\text{there exists a sequence } \{x_n\} \subset (x_0, y^*) \text{ such that}\\ &x_n \searrow x_0 \text{ and } f(x_n) > \eta \text{ for each } n \in \mathbb{N}.\end{aligned} \tag{5}$$

Let us consider the following cases:

I. $f(x_0) < \eta$. So we can analyze the following subcases:

I.1. If $\limsup_{n\to\infty} f(x_n) > \eta$, then there exists $\alpha \in \mathrm{L}^+(f, x_0)$ such that $\alpha > \eta$. Hence $\alpha > \eta > f(x_0)$ and $\alpha \in \mathrm{L}^+(f, x_0)$. So, from condition 1° of Definition 2, we have that $f^{-1}(\eta) \cap (x_0, y^*) \cap \mathcal{C}(f) \neq \emptyset$. So consequently

$$f^{-1}(\eta) \cap (x^*, y^*) \cap \mathcal{C}(f) \neq \emptyset.$$

This proves that the condition (2) holds, which contradicts our supposition.

I.2. Let $\lim_{n\to\infty} f(x_n) = \eta$. Hence $\eta \in \mathrm{L}^+(f, x_0)$. So, we can choose a subsequence $\{x_{k_n}\}$ of the sequence $\{x_n\}$ such that $x_{k_n} \searrow x_0$ and $f(x_{k_n}) \searrow \eta$.

Moreover, $f(x_0) < \eta$ and f is a Darboux function, which implies that

$$\text{there exists a sequence } \{y_m\} \text{ such that } y_m \searrow x_0 \text{ and } f(y_m) \nearrow \eta. \tag{6}$$

Since $x_0 \in \mathcal{S}^+(f)$, condition 2° of Definition 2 implies (2), which contradicts our supposition.

II. Now we suppose that $f(x_0) = \eta$. This implies that $x^* < x_0$. According to the (5) and the Darboux property of f, we can deduce that

$$\begin{gathered}\text{there exists a sequence } \{z_n\} \subset (x_0, y^*) \\ \text{such that } z_n \searrow x_0 \text{ and } f(z_n) \searrow \eta.\end{gathered} \tag{7}$$

Let us consider the following cases:

II.1. There exists a sequence $\{t_n\} \subset (x_0, y^*)$ such that $t_n \searrow x_0$ and $f(t_n) \nearrow \eta$. Then, according to (7), in a similar way as in I.2. we can prove that (2) holds, an impossibility.

II.2. There is no sequence $\{t_n\} \subset (x_0, y^*)$ such that $t_n \searrow x_0$ and $f(t_n) \nearrow \eta$. According to the above and our supposition that (2) does not hold, there exists $\sigma > 0$ such that $f((x_0, x_0 + \sigma) \cap \mathcal{C}(f)) \subset (\eta, +\infty)$. On the other hand, by the definition of x_0 and our supposition that (2) does not hold, we can infer $f((x^*, x_0) \cap \mathcal{C}(f)) \subset (-\infty, \eta)$. Hence x_0 cuts the function f. And again according to our supposition $x_0 \notin \mathcal{C}(f)$ and consequently $x_0 \notin \mathcal{S}(f)$, which is impossible. □

It is easy to prove the following fact.

Corollary 13. *Let $f : [a, b] \longrightarrow \mathbb{R}$. If a function $f \in \mathcal{S}$, then the functions $f|(a, b]$, $f|[a, b)$ and $f|(a, b)$ have the strong Świątkowski property.*

It is known that if a function $f : [a, b] \longrightarrow \mathbb{R}$ is quasi-continuous, the graph of $f|(\mathcal{C}(f))$ is a dense set in the graph of the function f ([2], [3]). For a function quasi-continuous at a single point the analogous theorem is not true. On the other hand, it seems to be interesting that the analogous fact holds for points at which the function f has the strong Świątkowski property (Theorem 15).

Proposition 14. *There exists a Darboux function $f : [a, b] \longrightarrow \mathbb{R}$ such that $\lambda(\mathcal{Q}_c(f)) > 0$ and $\mathcal{C}(f) = \emptyset$.*

PROOF. Let $\mathbf{C} \subset [a, b]$ be a Cantor set such that $a, b \in \mathbf{C}$ and $\lambda(\mathbf{C}) > 0$ ([13]). Let $\mathbf{I}$ be an arbitrary interval "removed" during the construction of $\mathbf{C}$ and let $*$ be the equivalence relation in $\mathbf{I}$ defined in the introduction. Denote by $\mathcal{A}_{\mathbf{I}}$ the set of all equivalence classes of this relation. Obviously $\operatorname{card}(\mathcal{A}_{\mathbf{I}}) = \mathfrak{c}$.

If $\mathbf{I}$ is an interval "removed" at an odd step of the construction of the Cantor set, there exists a bijection $k_{\mathbf{I}} : \mathcal{A}_{\mathbf{I}} \xrightarrow{onto} (0,1)$. Put $g_{\mathbf{I}}(x) = k_{\mathbf{I}}([x]_*)$ for $x \in \mathbf{I}$. Thus $g_{\mathbf{I}} : \mathbf{I} \longrightarrow (0,1)$ and for each nondegenerate interval $B \subset \mathbf{I}$, $g_{\mathbf{I}}(B) = (0,1)$. If $\mathbf{I}$ is an interval "removed" at the "n- step" (where n is an even number) of the construction of the Cantor set, there exists a bijection $s_{\mathbf{I}} : \mathcal{A}_{\mathbf{I}} \xrightarrow{onto} (0, \frac{1}{n})$. Put $h_{\mathbf{I}}(x) = s_{\mathbf{I}}([x]_*)$ for $x \in \mathbf{I}$. Thus $h_{\mathbf{I}} : \mathbf{I} \longrightarrow (0, \frac{1}{n})$ and for each nondegenerate interval $B \subset \mathbf{I}$ we have $h_{\mathbf{I}}(B) = (0, \frac{1}{n})$.

Define a function $f : [a,b] \longrightarrow [0,1)$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbf{C}, \\ g_{\mathbf{I}}(x), & \text{if } x \in \mathbf{I}, \text{ and } \mathbf{I} \text{ is an interval "removed" at an odd step,} \\ h_{\mathbf{I}}(x), & \text{if } x \in \mathbf{I}, \text{ and } \mathbf{I} \text{ is an interval "removed" at an even } n\text{-step.} \end{cases}$$

It is easy to see that $f \in \mathcal{D}bx$. Now, we shall prove that

$$\mathcal{Q}_c(f) = \mathbf{C}. \tag{8}$$

First, we will show the inclusion $\mathbf{C} \subset \mathcal{Q}_c(f)$. Let $\varepsilon > 0$ and $x_0 \in \mathbf{C}$. Then $f(x_0) = 0$. Let $\delta > 0$. There exists an interval $\mathbf{I}_0$ "removed" at the n_0th step (where n_0 is an even number) of the construction of the Cantor set, such that $n_0 > \frac{1}{\varepsilon}$ and $\mathbf{I}_0 \subset (x_0 - \delta, x_0 + \delta)$. Moreover, $f(\mathbf{I}_0) = h_{\mathbf{I}_0}(\mathbf{I}_0) = (0, \frac{1}{n_0}) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. So $x_0 \in \mathcal{Q}_c(f)$. To prove (8) it is sufficient to show that $\mathcal{Q}_c(f) \subset \mathbf{C}$. Let $y_0 \notin \mathbf{C}$. There exists an interval $\mathbf{I}^0$ "removed" at an nth step of the construction of Cantor set such that $y_0 \in \mathbf{I}^0$. Let $\delta^* > 0$ be positive number such that $(y_0 - \delta^*, y_0 + \delta^*) \subset \mathbf{I}^0$. Let U be an arbitrary nonempty, open set such that $U \subset (y_0 - \delta^*, y_0 + \delta^*)$.

Now let $\mathbf{I}^0$ be an interval "removed" at the n_0-step of the construction of Cantor set, where n_0 is an even number. (If $\mathbf{I}^0$ is an interval "removed" at an odd step, considerations are analogous.) Obviously $f(y_0) \in (0, \frac{1}{n_0})$. Let $\varepsilon_0 > 0$ be number such that $(f(y_0) - \varepsilon_0, f(y_0) + \varepsilon_0) \subsetneq (0, \frac{1}{n_0})$. From the fact that $f(U) = (0, \frac{1}{n_0})$ we deduce that $f(U) \not\subset (f(y_0) - \varepsilon_0, f(y_0) + \varepsilon_0)$. So $y_0 \notin \mathcal{Q}_c(f)$. The proof of (8) is complete.

From (8) we infer that $\lambda(\mathcal{Q}_c(f)) > 0$. Of course, $\mathcal{C}(f) = \emptyset$. □

In the case, when points belong to $\mathcal{S}(f)$ we have the following.

Theorem 15. *Let $f : [a,b] \to \mathbb{R}$ and let $x_0 \in \mathcal{S}(f)$. Then $(x_0, f(x_0)) \in \overline{\Gamma(f|\mathcal{C}(f))}$.*

PROOF. Let $f : [a,b] \to \mathbb{R}$ and $x_0 \in \mathcal{S}(f)$. If $x_0 \in \mathcal{C}(f)$, then obviously $(x_0, f(x_0)) \in \overline{\Gamma(f|\mathcal{C}(f))}$. Now, let $x_0 \in \mathcal{D}(f)$. Assume that $x_0 \neq b$ and $x_0 \in$

$\mathcal{D}^+(f)$. (If $x_0 \neq a$ and $x_0 \in \mathcal{D}^-(f)$ the proof is similar.) There exists $\alpha \in \mathrm{L}^+(f, x_0) \setminus \{f(x_0)\}$. Let us suppose, for instance, that $\alpha > f(x_0)$. Let $n_0 \in \mathbb{N}$ be a number such that $f(x_0) + \frac{1}{n_0} \in (f(x_0), \alpha)$ and $x_0 + \frac{1}{n_0} < b$. Then $\beta_n = f(x_0) + \frac{1}{n} \in (f(x_0), \alpha)$, for each $n \geq n_0$. From condition 1° of Definition 2 there exist $x_n \in (x_0, x_0 + \frac{1}{n}) \cap \mathcal{C}(f)$ (for $n \geq n_0$) such that $f(x_n) = \beta_n = f(x_0) + \frac{1}{n}$, for each $n \geq n_0$. Thus we have defined the sequence $\{x_n\} \subset \mathcal{C}(f)$ such that $x_n \longrightarrow x_0$ and $f(x_n) \longrightarrow f(x_0)$. So $(x_0, f(x_0)) \in \overline{\Gamma(f|\mathcal{C}(f))}$.

□

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