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A NOTE ON CLT GROUPS

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Let A, B, C be respectively the class of all finite supersolvable groups, the class of all finite groups which satisfy the converse to Lagrange's theorem, and the class of all finite solvable groups. We show that $A \subset B \subset C$, and give examples to show that both of the inclusions are actually proper.

Throughout, 'n', 't', 'a_1', 'a_2', $\cdots$, 'a_t' will denote positive integers; 'p_1', 'p_2', $\cdots$, 'p_t' will denote pairwise distinct positive integer primes. If G and H are finite groups, then, 'G'' will denote the commutator subgroup of G, '$G \times H$' will denote the external direct product of G and H, and '$|G|$' will denote the order of G. 'A_4' will denote the alternating group on 4 symbols, 'e' will denote the identity of A_4, and 'C_2' will denote the cyclic group of order 2.

We are concerned here only with finite groups; throughout, when we say 'group', we intend this to be read as 'finite group', and 'G' will always denote a finite group. Our version of the converse to Lagrange's theorem is as follows:

DEFINITION. *G is a CLT group* if and only if for each d, the following holds: if d is a positive integer divisor of $|G|$, then G has at least one subgroup H with $|H| = d$.

All terminology not used in the above definition will be that of [2].

LEMMA 1. $|G| = n = p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$ *and* $n_i = n/p_i^{a_i}$ *for* $i = 1, 2, \cdots, t$. *Then G is solvable if and only if G has subgroups with orders* $n_1, n_2, \cdots, n_t$.

Proof. This follows readily from Theorem 9.3.1, p. 141, and Theorem 9.3.3, p. 144 of [2].

LEMMA 2. $|G| = n = p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$ *with* $p_1 < p_2 < \cdots < p_t$. *Then if G is supersolvable, G has normal subgroups with orders* $1, p_t, p_t^2, \cdots, p_t^{a_t}$.

Proof. This follows readily from Corollary 10.5.2, p. 159 of [2].

THEOREM 1. *Every CLT group is solvable.*

Proof. This is trivial if $|G| = 1$. Let G be a CLT group with $|G| = n = p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$, and let $n_i = n/p_i^{a_i}$ for $i = 1, 2, \cdots, t$; since each n_i is a divisor of $|G|$, G must have subgroups with orders n_1, $n_2, \cdots, n_t$. Applying Lemma 1, we conclude that G is solvable, and this completes our proof.

The author wishes to thank Professor M. Hall for pointing out the proof of Theorem 1.

THEOREM 2. *Every supersolvable group is CLT.*

Proof. This is trivial if $|G| = 1$. We shall use induction on the number of positive integer primes dividing $|G|$ if $|G| > 1$.

If G is *any* group with $|G| = p_1^{a_1}$, then Sylow's theorem tells us that G is CLT; in fact, any finite p_1-group is supersolvable, but we do not need this.

Suppose now that every supersolvable group whose order is divisible by exactly t distinct positive integer primes is CLT, and let G be a supersolvable group with $|G| = p_1^{a_1}p_2^{a_2} \cdots p_t^{a_t}p_{t+1}^{a_{t+1}}$, $p_1 < p_2 < \cdots < p_t < p_{t+1}$. We shall show that G is CLT, and our conclusion will follow. Let d be a positive integer divisor of $|G|$; we wish to show that G has a subgroup of order d. We may write $d = p_1^{b_1}p_2^{b_2} \cdots p_t^{b_t}p_{t+1}^{b_{t+1}} = rp_{t+1}^{b_{t+1}}$, where b_i is an integer and $0 \leqq b_i \leqq a_i$ for each $i = 1, 2, \cdots, t, t+1$, and $r = p_1^{b_1}p_2^{b_2} \cdots p_t^{b_t}$. Since G is supersolvable, G is solvable, and we may apply Lemma 1 to conclude that G has a subgroup H with $|H| = n_{t+1} = p_1^{a_1}p_2^{a_2} \cdots p_t^{a_t}$. Now, H is a subgroup of G, and G is supersolvable; hence, H is supersolvable and $|H|$ is divisible by exactly t distinct positive integer primes. By our induction hypothesis, H is CLT; since r is a divisor of n_{t+1} and $n_{t+1} = |H|$, it follows that H must have a subgroup R with $|R| = r$. Thus, R is a subgroup of G with $|R| = r$; since G is supersolvable and p_{t+1} is the largest prime dividing $|G|$, we may apply Lemma 2 to conclude that G has a *normal* subgroup P with $|P| = p_{t+1}^{b_{t+1}}$. Now let RP be the set of all products xy with $x \in R$ and $y \in P$; since P is a *normal* subgroup of G, RP is a subgroup of G. Also, $|R|$ and $|P|$ are relatively prime, so that $|RP| = |R|\cdot|P|/|R \cap P| = |R|\cdot|P|$; hence, RP is a subgroup of G with $|RP| = |R|\cdot|P| = rp_{t+1}^{b_{t+1}} = d$, and this completes our proof.

REMARK. Since every subgroup of a supersolvable group is supersolvable, it is clear that Theorem 2 can be used to prove the following: If G is supersolvable, then *every* subgroup of G is CLT. Sometime after the author had obtained Theorem 2, he became aware of the following (due to Professor W. Deskins): G is supersolvable if and only if *every* subgroup of G (including G itself) is CLT. This appears in [1].

LEMMA 3. *Let H be any group with $|H| = h$, where h is odd.*

Then $|A_4 \times H| = 12h$, *and* $A_4 \times H$ *has no subgroups of order* $6h$.

Proof. Suppose to the contrary that $A_4 \times H$ has a subgroup K with $|K| = 6h$; then K has index 2 in $A_4 \times H$, so that K is a normal subgroup of $A_4 \times H$ and $|(A_4 \times H)/K| = 2$. Hence, $(A_4 \times H)/K$ is Abelian, so that $(A_4 \times H)' = A_4' \times H'$ is a subgroup of K; it follows that $|A_4'|$ must divide $|K|$. Now $A_4' = \{e, (12)(34), (13)(24), (14)(23)\}$, so that 4 must divide $|K| = 6h$; this is not possible, since h is odd, and this completes our proof.

LEMMA 4. *Let* H *be any group of odd order*; *then* $A_4 \times H$ *is solvable and not* CLT.

Proof. According to Thompson and Feit, H is solvable; since A_4 is solvable, it follows that $A_4 \times H$ is solvable. The result of Lemma 3 shows that $A_4 \times H$ is not CLT, and this completes our proof.

LEMMA 5. *Let* G *be any* CLT *group*; *then* $(A_4 \times C_2) \times G$ *is* CLT *and not supersolvable*.

Proof. It is clear that a finite direct product of CLT groups is itself CLT, and it is clear that $A_4 \times C_2$ is CLT; it follows that $(A_4 \times C_2) \times G$ is CLT. Now Lemma 3 shows that A_4 is not CLT, and Theorem 2 then shows that A_4 is not supersolvable; it follows that $(A_4 \times C_2) \times G$ is not supersolvable, and this completes our proof.

Our results show that the class of CLT groups fits properly between the class of supersolvable groups and the class of solvable groups. As a closing remark, we note the following: If G is supersolvable (solvable) then every subgroup of G and every factor group of G is supersolvable (solvable); that this is not true for CLT groups in general is shown by the following example. Let M be any CLT group; then $(A_4 \times C_2) \times M$ is CLT, but $(A_4 \times C_2) \times M$ has A_4 as both a subgroup and a factor group, and A_4 is not CLT.

REFERENCES

1. W. E. Deskins, *A characterization of finite supersolvable groups*, Amer. Math. Monthly, **75** (1968), 180-182.

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