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DERIVATIONS OF C^*-ALGEBRAS HAVE SEMI-CONTINUOUS GENERATORS

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For each derivation δ of a C^*-algebra A with $\delta(x^*) = -\delta(x)^*$ there exists a minimal positive element h in the enveloping von Neumann algebra A'' such that $\delta(x) = hx - xh$. It is shown that the generator h belongs to the class of lower semi-continuous elements in A''. From this it follows that if the function $\pi \to \|\pi \circ \delta\|$ is continuous on the spectrum of A then h multiplies A. This immediately implies that each derivation of a simple C^*-algebra is given by a multiplier of the algebra. Another application shows that each derivation of a countably generated monotone sequentially closed C^*-algebra is inner.

A linear operator δ on a C^*-algebra A is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all a and b in A. If $\delta^* = -\delta$ (i.e., $\delta(a)^* = -\delta(a^*)$) then $\alpha_t(a) = \exp(it\delta)a$ defines a norm-continuous one-parameter group of $*$-automorphisms of A. Conversely, each such group can be written as $\exp(it\delta)$ for a suitable derivation δ of A. After a number of partial results, notably by I. Kaplansky and R. V. Kadison, it was proved by S. Sakai that every derivation of a von Neumann algebra A is inner, i.e., $\delta(a) = ha - ah$ for some h in A (see [9, III.9.3. Théorème 1]). Recently W. B. Arveson ([3])—see also [4]—gave a new proof of this result, using the theory of spectral subspaces associated with a one-parameter group of automorphisms. The powerful techniques developed in [3] enabled the first author to show that each derivation of an AW^*-algebra is inner ([12]).

In this paper we use Arveson's technique to show that if δ is a derivation of a C^*-algebra A with $\delta^* = -\delta$ then the minimal positive generator for δ, or rather for its extension to a derivation of the enveloping von Neumann algebra A'' of A, is the limit of an increasing net of self-adjoint operators from $\tilde{A}$. This shows that the function $\pi \to \|\pi \circ \delta\|$ on the spectrum $\hat{A}$ of A is lower semi-continuous and that it is continuous if and only if the minimal positive generators for δ and $-\delta$ both multiplies A. This last result was first proved in [2] and has as an immediate consequence that every derivation of a simple C^*-algebra is given by a multiplier ([17]). We finally show that every derivation of a countably generated monotone sequentially closed C^*-algebra is inner.

The possibility of using [12] to show that derivations of C^*-algebras have measurable generators was pointed out to us by E. B. Davies.

1. Spectral subspaces and duality. Let α_t be a norm-continuous one-parameter group of isometries of a Banach space X. For each f in $L^1(\boldsymbol{R})$ let $\pi_\alpha(f)$ denote the bounded operator on X given by the Bochner integral

$$\pi_\alpha(f)x = \int \alpha_t(x)f(t)dt .$$

With $\hat{f}(s) = \int f(t)e^{ist}dt$ and $-\infty \leqq t \leqq w \leqq \infty$ let $R_\alpha(t, w)$ denote the closed subspace of X generated by vectors $\pi_\alpha(f)x$, $x \in X$ such that $\hat{f}$ has compact support in (t, w). The spectral subspace associated with $[t, w]$ is

$$M_\alpha[t, w] = \bigcap R_\alpha\left(t - \frac{1}{n}, w + \frac{1}{n}\right).$$

As shown in [3, Proposition 2.2]—see also [12]—we have

$$M_\alpha[t, w] = \{x \in X \mid \pi_\alpha(f)x = 0 \ \forall f \in I_0[t, w]\}$$

where $I_0[t, w]$ denotes the set of functions f in $L^1(\boldsymbol{R})$ such that $\hat{f}$ has compact support disjoint from $[t, w]$.

The transposed α_t^* and bi-transposed α_t^{**} of α_t gives rise to norm-continuous (and weak *-continuous) groups of isometries of X^* and X^{**}, respectively. We shall relate the spectral subspaces of the three groups using polar sets (denoted by M^0).

LEMMA 1.1. *If $s < t$ then*

$$M_{\alpha^*}[-\infty, s] \subset M_\alpha[t, \infty]^0 \subset M_{\alpha^*}[-\infty, t] .$$

Proof. For each f in $I_0[-\infty, t]$ and x in X we have $\pi_\alpha(f)x \in R_\alpha(t, \infty)$. If therefore $\rho \in R_\alpha(t, \infty)^0$ then

$$0 = \langle \pi_\alpha(f)x, \rho \rangle = \langle x, \pi_{\alpha^*}(f)\rho \rangle ,$$

since $\pi_{\alpha^*}(f)$ is the transposed of $\pi_\alpha(f)$. Thus $\pi_{\alpha^*}(f)\rho = 0$ so that $\rho \in M_{\alpha^*}[-\infty, t]$. It follows that

$$R_\alpha(t, \infty)^0 \subset M_{\alpha^*}[-\infty, t] ,$$

and

$$R_\alpha(t, \infty) \subset M_\alpha[t, \infty]$$

implies that

$$M_\alpha[t, \infty]^0 \subset R_\alpha(t, \infty)^0 .$$

Consequently

$$M_\alpha[t, \infty]^0 \subset M_\alpha*[-\infty, t] .$$

If $s < t$ then $s < t - (1/n)$ for sufficiently large n. For each f in $L^1(\boldsymbol{R})$ where $\hat{f}$ has compact support in $(t-(1/n), \infty)$ and x in X we have

$$\langle \pi_\alpha(f)x, \rho \rangle = \langle x, \pi_\alpha*(f)\rho \rangle = 0$$

for each ρ in $M_\alpha*[-\infty, s]$, since $f \in I_0[-\infty, s]$. Thus $\rho \in R_\alpha(t-(1/n), \infty)^0$ and a fortiori $\rho \in M_\alpha[t, \infty]^0$. It follows that

$$M_\alpha*[-\infty, s] \subset M_\alpha[t, \infty]^0$$

and the proof is complete.

REMARK 1.2. The reader may verify that for each x in $M_\alpha[t, \infty]$ and ρ in $M_\alpha*[-\infty, t]$ one has $\langle \alpha_s(x), \rho\rangle = e^{ist}\langle x, \rho\rangle$ for all s. Despite this extraordinary behavior it is not in general true that $M_\alpha[t, \infty]^0 = M_\alpha*[-\infty, t]$. To see this take any Banach space X and define $\alpha_t(x) = e^{it}x$ for all x in X. Then α_t is a norm-continuous one-parameter group of isometries of X. Since $\pi_\alpha(f)x = \hat{f}(1)x$ it is easily verified that $M_\alpha[t, \infty] = X$ for $t \leqq 1$ and zero otherwise. Analogously $M_\alpha*[-\infty, t] = X^*$ for $t \geqq 1$ and zero otherwise. Consequently,

$$0 = M_\alpha[1, \infty]^0 \neq M_\alpha*[-\infty, 1] = X^* .$$

PROPOSITION 1.3. *If $s < t$ then*

$$M_\alpha**[t, \infty] \subset M_\alpha[s, \infty]^{00} \subset M_\alpha**[s, \infty] .$$

Proof. Taking polar sets in Lemma 1.1 we get

$$(*) \qquad M_\alpha*[-\infty, s]^0 \subset M_\alpha[s, \infty]^{00} \subset M_\alpha*[-\infty, w]^0$$

for $w < s$. Using Lemma 1.1 with α^*_{-t} and X^* instead of α_t and X we obtain

$$M_\alpha**[t, \infty] \subset M_\alpha*[-\infty, s]^0 \quad \text{and} \quad M_\alpha*[-\infty, w]^0 \subset M_\alpha**[w, \infty] ,$$

for $s < t$. Inserting these inclusions in (*) yield

$$M_\alpha**[t, \infty] \subset M_\alpha[s, \infty]^{00} \subset M_\alpha**[w, \infty]$$

for $w < s < t$. However, by the definition of spectral subspaces

$$M_\alpha**[s, \infty] = \bigcap_{w<s} M_\alpha**[w, \infty]$$

and the proposition follows.

2. Derivations of C^*-algebras. Let A be a C^*-algebra and denote by A'' the enveloping von Neumann algebra of A, isomorphic with the second dual of A (see [7, § 12]). For any set B in $A''_{s.a.}$ let B^- denote the norm-closure of B and let B^m denote the set of operators in $A''_{s.a.}$ which can be obtained as strong limits of increasing nets from B. The class $((A_{s.a.})^m)^-$ consists of the so called lower semi-continuous elements of $A''_{s.a.}$. If $A_{s.a.}$ is represented as the continuous real affine functions vanishing at 0 on the convex compact set

$$Q = \{\rho \in A^* \mid \|\rho\| \leqq 1,\ \rho \geqq 0\}$$

then $((A_{s.a.})^m)^-$ is precisely the set of lower semi-continuous bounded real affine functions on Q vanishing at zero. Let $\tilde{A}$ denote the C^*-algebra obtained by adjoining the unit 1 of A'' to A. Then

$$((A_{s.a.})^m)^- + \boldsymbol{R}1 = (\tilde{A}_{s.a.})^m .$$

If $M(A)$ denotes the C^*-algebra in A'' of elements x such that $xA \subset A$ and $Ax \subset A$ then

$$M(A)_{s.a.} = (\tilde{A}_{s.a.})^m \cap (\tilde{A}_{s.a.})_m .$$

([15, Theorem 2.5] see also [1]). It is shown in [8, Theorem 5] (see also [15, Corollary 4.7]) that the center of $M(A)$—the ideal center of A—can be identified with the set of bounded continuous functions on the spectrum $\hat{A}$ of A.

Let δ be a derivation of A such that $\delta^* = -\delta$. Then $\alpha_t = \exp it\delta$ defines a norm-continuous one-parameter group of $*$-automorphisms of A so that the results from § 1 are applicable.

THEOREM 2.1. *Let δ be a derivation of a C^*-algebra A such that $\delta^* = -\delta$. Then the minimal positive operator h in A'' for which $\delta = \operatorname{ad} h$ is a lower semi-continuous element.*

Proof. The bi-transposed $\bar{\bar{\delta}}$ of δ is an extension of δ to a derivation of A''. With $\bar{p}(t)$ as the left annihilator projection of $M_{\alpha^{**}}[t, \infty]$ we know from [12, Proposition 3] that the operator-valued Riemann-Stieltjes integral

$$\int_0^{\|\delta\|} t d\bar{p}(t)$$

with respect to the increasing projection-valued map $t \rightarrow \bar{p}(t)$ defines a positive operator h in A'' and that h is the minimal positive operator in A'' such that $\bar{\bar{\delta}} = \operatorname{ad} h$.

Let $p(t)$ denote the left annihilator projection in A'' of $M_\alpha[t, \infty]$. Since the annihilators of a subspace and its weak closure (=bi-polar) coincide we see from Proposition 1.3 that

$$\bar{p}(s) \leqq p(s) \leqq \bar{p}(t)$$

for $s < t$. For each positive functional ρ on A'' define g and $\bar{g}$ on $[0, \|\delta\|]$ by

$$g(t) = \rho(p(t)) \quad \text{and} \quad \bar{g}(t) = \rho(\bar{p}(t)) .$$

Since $\bar{g}(s) \leqq g(s) \leqq \inf \bar{g}(t)$ it follows from well-known properties of Riemann-Stieltjes integrals that

$$\int f(t)d\bar{g}(t) = \int f(t)dg(t)$$

for every continuous function f on $[0, \|\delta\|]$.

Thus

$$\rho\Big(\int_0^{\|\delta\|} f(t)d\bar{p}(t)\Big) = \rho\Big(\int_0^{\|\delta\|} f(t)dp(t)\Big) ,$$

and since this holds for all ρ on A'' we have

$$\int_0^{\|\delta\|} f(t)d\bar{p}(t) = \int_0^{\|\delta\|} f(t)dp(t) .$$

In particular

$$h = \int_0^{\|\delta\|} t\, dp(t) .$$

For fixed t let Λ denote the net (under inclusion) of finite subsets of $M_\alpha[t, \infty]$, and for λ in Λ let $|\lambda|$ denote the cardinality of λ. Then the net in A_+ with elements

$$x_\lambda = \Big(|\lambda|^{-1} + \sum_{x \in \lambda} xx^*\Big)^{-1} \sum_{x \in \lambda} xx^*$$

increases to a projection $q(t)$ in $(A_{s.a.})^m$. Since

$$q(t) \geqq (|\lambda|^{-1} + xx^*)^{-1}xx^*$$

for each x in λ we see that $q(t)$ majorizes the range projection of each x in $M_\alpha[t, \infty]$. Thus if H is the universal Hilbert space on which A'' acts we conclude that $q(t)$ is the projection on the closure of $M_\alpha[t, \infty]H$. It follows that $q(t) = 1 - p(t)$. Put

$$h_n = n^{-1} \sum_{k=1}^{n} q(kn^{-1} \|\delta\|) .$$

Then $h_n \in (A_{s.a.})^m$ and $0 \leqq h - h_n \leqq n^{-1}$; so that $h \in ((A_{s.a.})^m)^-$ which is precisely what we wanted.

PROPOSITION 2.2. *For each derivation δ of a C^*-algebra A the function $\pi \longrightarrow \|\pi \circ \delta\|$ is lower semi-continuous on $\hat{A}$.*

Proof. Given π_0 in $\hat{A}$ let t be the lim inf of $\|\pi \circ \delta\|$ when π ranges over the neighborhood system of π_0. We shall prove that $\|\pi_0 \circ \delta\| \leqq t$. Choose a net $\{\pi_i\}$ in $\hat{A}$ converging to π_0 such that $\|\pi_i \circ \delta\| < t + \varepsilon$ for all i. Then

$$\bigcap \ker \pi_i \subset \ker \pi_0 .$$

If therefore ρ denotes the representation $\Sigma^{\oplus}\pi_i$ then $\pi_0(A)$ is a quotient of $\rho(A)$ so that $\|\pi_0 \circ \delta\| \leqq \|\rho \circ \delta\|$. But

$$\|\rho \circ \delta\| = \sup \|\pi_i \circ \delta\| \leqq t + \varepsilon$$

and consequently $\|\pi_0 \circ \delta\| \leqq t + \varepsilon$. Since $\varepsilon > 0$ is arbitrary the proposition follows.

REMARK 2.3. If $\delta^* = -\delta$ and h is the minimal positive generator for δ then since each representation π of A is quasi-equivalent to a representation of the form $x \longrightarrow zx$ for some central projection z in A'' we have

$$\|\pi \circ \delta\| = \|\bar{\delta} \mid A''z\| = \|hz\| = \|\pi(h)\| .$$

Note that since $h \in ((A_{sa})^m)^-$ the function $\pi \longrightarrow \|\pi(h)\|$ is lower semi-continuous on $\hat{A}$ by [15, Theorem 4.6] in accordance with Proposition 2.2.

The next result is proved in [2] by an entirely different method.

THEOREM 2.4. *For each derivation δ of a C^*-algebra A such that $\delta^* = -\delta$, the function $\pi \longrightarrow \|\pi \circ \delta\|$ is continuous on $\hat{A}$ if and only if the minimal positive generators for δ and $-\delta$ both belong to $M(A)$.*

Proof. Without loss of generality assume that $\|\delta\| = 1$, and let h and k be the minimal positive generators for δ and $-\delta$, respectively. Since $1 - k$ is a positive generator for δ we have $h \leqq 1 - k$. Moreover, $(1 - k) - h$ belongs to the center of A''. Put $a = h + k$. We claim that $\|\pi(h)\| = \|\pi(a)\|$ for each irreducible representation π of A. For if $\|\pi(h)\| + \varepsilon \leqq \|\pi(a)\|$ for some $\varepsilon > 0$ then since $\pi(a)$ is a multiple of the identity we get

$$\pi(a) - \pi(k) = \pi(h) \leqq \pi(a) - \varepsilon$$

so that $\varepsilon \leqq \pi(k)$. But this is impossible as $\pi(k)$ is the minimal positive generator for $-\pi \circ \delta$.

By the Dauns-Hofmann Theorem the central, positive element a

in $((A_{s.a.})^m)^-$ belongs to $M(A)$ if and only if the function $\pi \rightarrow \|\pi(a)\|$ $(=\|\pi \circ \delta\|)$ is continuous on $\hat{A}$ (see [15, Corollary 4.7]). If both h and k belongs to $M(A)$ then of course $a \in M(A)$. But if $a \in M(A)$ then in particular $a \in (\tilde{A}_{s.a.})_m$ and since $-(\tilde{A}_{s.a.})^m = (\tilde{A}_{s.a.})_m$

$$h = a - k \in (\tilde{A}_{s.a.})_m .$$

Thus by Theorem 2.1

$$h \in (\tilde{A}_{s.a.})^m \cap (\tilde{A}_{s.a.})_m = M(A)_{s.a.} .$$

This completes the proof.

COROLLARY 2.5. (Sakai [17]). *Every derivation of a simple C^*-algebra is given by a multiplier of the algebra.*

Proof. Each nonzero representation of A is an isometry so that the function $\pi \rightarrow \|\pi \circ \delta\|$ is constant, hence continuous.

3. Derivations of sequentially closed C^*-algebras. A monotone sequentially closed C^*-algebra B is a C^*-algebra in which every norm-bounded increasing sequence of self-adjoint elements has a least upper bound in the algebra. Basically these algebras are the non-commutative algebraic counterpart of abstract measure spaces, a point of view which has been successfully exploited in [5]. A monotone sequentially closed C^*-algebra which admits a faithful σ-normal representation on a Hilbert space (sometimes known as a Baire* algebra) is a reasonable non-commutative analogue of the Baire functions on a locally compact space. These algebras are studied in [6], [11], [13], [14], and [16].

We say that the monotone sequentially closed C^*-algebra B is countably generated if it contains a sequence $\{b_n\}$ such that the smallest monotone sequentially closed C^*-subalgebra of B containing $\{b_n\}$ is equal to B. In this case B has a unit—the supremum of all range projections of the b_n's.

THEOREM 3.1. *Every derivation of a countably generated monotone sequentially closed C^*-algebra is inner.*

Proof. We may assume that $\delta^* = -\delta$. Let A be the separable C^*-subalgebra of B generated by elements of the form $\delta^m(b_n)$, $m \geqq 0$, where $\{b_n\}$ is a generating sequence for B containing 1. Then $\delta(A) \subset A$ so that $\delta_1 = \delta \mid A$ is a derivation of A. By Theorem 2.1 $\delta_1 = ad\,h$ for some h in $(A_+)^m$ (since $1 \in A$ the subset $(A_{s.a.})^m$ is norm-closed and $((A_{s.a.})^m)_+ = (A_+)^m$). The separability of A implies that Q is metrizable so that we can find a sequence $\{h_k\}$ in A_+ with $h_k \nearrow h$.

Let $\{u_n\}$ be a countable group of unitaries in A which generate

A as a C^*-algebra. Note that

$$u_n^* h u_n - h = u_n^* \delta_1(u_n) \in A \ .$$

For fixed n_0 put $X = \sum_{n<n_0}^{\oplus} A$. Then the sequence in X with elements

$$x_k = \sum_{n \leqq n_0} (u_n^* h_k u_n - h_k - u_n^* \delta_1(u_n))$$

converges weakly to zero in X^{**}. Thus for every $\varepsilon > 0$ and k_0 there exists $\{x_k \mid k_0 \leqq k \leqq k_1\}$ such that

$$\bigcap_k \{\rho \in X_1^* \mid |\rho(x_k)| \geqq \varepsilon\} = \varnothing \ .$$

It follows that $\| \Sigma \lambda_k x_k \| < \varepsilon$ for some convex combination of the x_k's. Using this we can inductively find a sequence $\{a_m\}$ in A_+ such that

(i) Each a_m is a convex combination of elements from $\{h_k\}$.

(ii) The elements h_k occuring in the combination of a_{m+1} all have higher index than those occuring in a_m.

(iii) $\| u_n^* a_m u_n - a_m - u_n^* \delta_1(u_n) \| \leqq \dfrac{1}{m}$ for $n \leqq m$.

By condition (i) $a_m \leqq \|h\|$ for all m and by condition (ii) the sequence $\{a_m\}$ is increasing. Let a denote the least upper bound of $\{a_m\}$ in B. Then $u_n^* a u_n$ is the least upper bound for $\{u_n^* a_m u_n\}$. Since $\{u_n^* a_m u_n - a_m\}$ is norm-convergent to $u_n^* \delta_1(u_n)$ we conclude from [10, Lemma 2.2] that

$$u_n^* a u_n - a = u_n^* \delta_1(u_n)$$

for all u_n (the additional hypothesis in [10, Lemma 2.2] that B is (unrestrictedly) monotone complete is not needed for the proof). Thus $\delta_1(u_n) = a u_n - u_n a$ for all u_n. The elements in B on which the two derivations δ and $ad\, a$ coincide form a C^*-algebra containing A. Since $\delta(A) \subset A$ we see that the elements in B on which δ^n and $(ad\, a)^n$ coincide for every n form a C^*-algebra B_0 containing A. If $\{c_n\}$ is an increasing sequence of self-adjoint elements in B_0 with least upper bound c in B then

$$\exp(it\delta)c_n = \exp(ita)c_n \exp(-ita)$$

for every n and all real t. Since * automorphisms are order-preserving this implies that

$$\exp(it\delta)c = \exp(ita)c \exp(-ita) \ .$$

Successive differentiations show that $\delta^n(c) = (ad\, a)^n(c)$; hence $c \in B_0$. It follows that B_0 is monotone sequentially closed and therefore $B_0 = B$. This completes the proof.

COROLLARY 3.2. *If δ is a derivation of a countably generated monotone sequentially closed C^*-algebra B such that $\delta^* = -\delta$ then there is a minimal positive generator a in B for δ characterized by*

$$\| az \| = \| \delta \mid Bz \|$$

for every central projection z in B.

Proof. Since B is countably generated every projection in B has a central cover in B, so that B is well supplied with central projections. With the notation as in the proof of Theorem 3.1 note that each central projection z in B determines a representation π of A given by $\pi(b) = bz$. Since h is the minimal positive generator for δ_1 this implies that $\| \pi(h) \| = \| \pi \circ \delta_1 \|$. Now $0 \leqq a_m \leqq h$ so that

$$\| a_m z \| = \| \pi(a_m) \| \leqq \| \pi(h) \| .$$

Since az is the least upper bound in B of $\{a_m z\}$ we conclude that $0 \leqq az \leqq \| \pi(h) \|$, hence $\| az \| \leqq \| \pi(h) \|$. Finally

$$\| \pi \circ \delta_1 \| = \| \delta \mid Az \| \leqq \| \delta \mid Bz \|$$

so that $\| az \| \leqq \| \delta \mid Bz \|$. The reverse inequality is obvious and it follows as in the proof of [12, Proposition 3] that a is uniquely characterized by these norm conditions and that it is the minimal positive generator for δ in B.

REMARK 3.3. For a nonseparable Hilbert space H let $S(H)$ denote the set of operators in $B(H)$ with separable range. Then $S(H)$ is a monotone sequentially closed C^*-algebra but since $S(H)$ is an ideal in $B(H)$ it is easy to find outer derivations for $S(H)$. Thus the condition of being countably generated can not be deleted from Theorem 3.1.

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