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# ON THE IMPOSSIBILITY OF OBTAINING $S^2 \times S^1$ BY ELEMENTARY SURGERY ALONG A KNOT

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**Elementary surgery along a knot has been used in an attempt to construct a counterexample to the Poincaré Conjecture. Certain classes of knots have been examined, but no counterexample has yet been found. Another, and perhaps as interesting a question, is whether $S^2 \times S^1$ can be obtained by elementary surgery along a knot. In this paper the question is answered in the negative for knots with nontrivial Alexander polynomial, for composite knots, and for a large class of knots with trivial Alexander polynomial—the simply doubled knots.**

By a knot we will mean a polygonal simple closed curve in the 3-sphere $S^3$. A solid torus $T$ is a 3-manifold homeomorphic to $S^1 \times D^2$. The boundary of $T$ is a torus, a 2-manifold homeomorphic to $S^1 \times S^1$. A meridian of $T$ is a simple closed curve on Bd $T$ which bounds a disk in $T$ but is not homologous to zero on Bd $T$. A meridianal disk of $T$ is a disk $D$ in $T$ such that $D \cap \mathrm{Bd}\, T = \mathrm{Bd}\, D$, and $\mathrm{Bd}\, D$ is a meridian of $T$. A longitude of $T$ is a simple closed curve on Bd $T$ which is transverse to a meridian of $T$ and is null-homologous in $\overline{S^3 - T}$.

The basic construction, elementary surgery along a knot, is now described: Let $N$ be a regular neighborhood of a knot $K$, $m$ an oriented meridianal curve on Bd $N$, and $l$ an oriented curve on Bd $N$ which is transverse to $m$ and bounds an orientable surface in $\overline{S^3 - N}$. Let $T$ be a solid torus and let $h: T \rightarrow N$ be a homeomorphism. Then $S^3$ is homeomorphic to $\overline{S^3 - N} \cup_{h|\mathrm{Bd}\,T} T$. Now let $h_1: \mathrm{Bd}\, T \rightarrow \mathrm{Bd}\, N$ be a homeomorphism with the property that $h^{-1} \cdot h_1: \mathrm{Bd}\, T \rightarrow \mathrm{Bd}\, T$ does not extend to a homeomorphism of $T$ onto $T$. Let $M^3 = \overline{S^3 - N} \cup_{h_1} T$, then we say that $M^3$ is obtained from $S^3$ by performing an elementary surgery along $K$.

Consider now the fundamental group of the complement of the knot $\pi_1(\overline{S^3 - N})$ with base point $m \cap l$, where $m$ and $l$ are considered as elements of $\pi_1(\overline{S^3 - N}) = G$. Then the coset $\bar{m} = mG'$ generates the commutator quotient group $G/G' = H_1(\overline{S^3 - N})$, and the longitude $l$ is in the second commutator subgroup $G''$. The fundamental group of $M^3$ is obtained by adjoining the relation $l^p = m^q$ to $\pi_1(\overline{S^3 - N})$ where $pl - qm$ is the image under $h_1$ of the boundary of a meridianal disk of $T$, $p$ and $q$ are relatively prime, and $p > 0$. The first homology group of $M^3$ is generated by $\bar{m}$ with the relation $\bar{m}^q = 1$.

Thus if $M^3$ is homeomorphic to $S^2 \times S^1$, then $\pi_1(M^3) \simeq H_1(M^3) \simeq Z$. Hence, $q = 0$ and $p = 1$; that is, a longitudinal surgery is performed in which the image of the boundary of a meridianal disk is a longitude. It should be noted that a longitudinal surgery along a trivial knot does yield $S^2 \times S^1$. In the following theorem we give a necessary condition that a surgered manifold be homeomorphic to $S^2 \times S^1$.

THEOREM 1. *If a manifold homeomorphic to $S^2 \times S^1$ results from elementary surgery along a knot $K$, then the Alexander polynomial of $K$ is trivial.*

*Proof.* If a surgered manifold $M^3$ is homeomorphic to $S^2 \times S^1$, then a longitudinal surgery must have been performed. The fundamental group of $M^3$ is obtained by adding the relation $l = 1$ to $\pi_1(\overline{S^3 - N}) = G$. In other words, $\pi_1(M^3)$ is the quotient group of $G$ by the normal closure of the subgroup generated by $l$; denote this subgroup by $(l)^c$. Now since $l \in G''$ and $G''$ is a characteristic subgroup of $G'$, it follows that $(l)^c \leqq G'' \leqq G'$. Thus if $G''$ is a proper subgroup of $G'$, then $\pi_1(M^3) \neq Z$ and $M^3$ is not homeomorphic to $S^2 \times S^1$. But $G''$ is a proper subgroup of $G'$ if and only if the Alexander polynomial of $K$ is nontrivial [1]. This establishes Theorem 1.

So now we consider a large class of nontrivial knots with trivial Alexander polynomial—the simply doubled knots. A simply doubled knot or a doubled knot without twists is defined as follows: Let $T_0$ be a standardly embedded solid torus in $S^3$ with meridian $m_0$ and longitude $l_0$. Let $J$ be a self-linking simple closed curve in $T_0$ (as shown in Figure 1 for the trefoil) and let $T_1$ be a regular neighborhood of $J$ in $T_0$ with meridian $m_1$ and longitude $l_1$. Let $K$ be a nontrivial knot in $S^3$, $N(K)$ a regular neighborhood of $K$ with meridian $m$ and longitude $l$ which bounds an orientable surface in $\overline{S^3 - N(K)}$. Let $f\colon T_0 \to N(K)$ be a homeomorphism with the property that $f(m_0) = m$ and $f(l_0) = l$, then we say that $K$ is simply doubled to obtain $f(J)$.

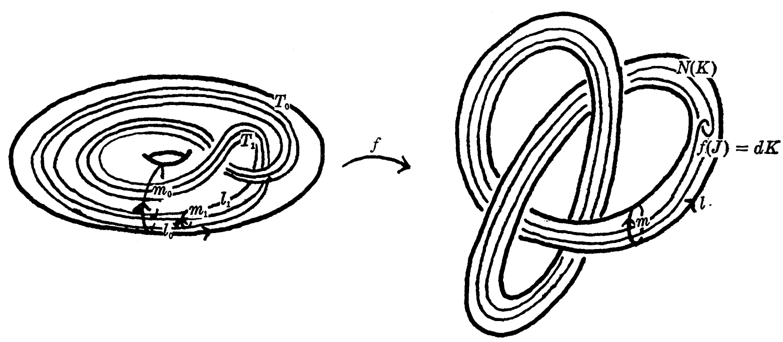


FIGURE 1.

The doubled knot $f(J)$ we will denote by $dK$.

Consider now the fundamental group of $\overline{T_0 - T_1}$ with base point $m_0 \cap l_0$; let $G_1 = \pi_1(\overline{T_0 - T_1})$ and let $G(K) = \pi_1(\overline{S^3 - N(K)})$. By van Kampen's theorem, the group of the double of $K$, $G(dK) = \pi_1(\overline{S^3 - N(dK)})$, is the free product with amalgamation $G(K)*G_1$ with the identification of subgroups $(l, m)$ of $G(K)$ and $(l_0, m_0)$ of $G_1$ determined by $l = l_0$ and $m = m_0$. Furthermore, $G_1$ is generated by $l_0$ and $m_1$ subject to the relation $[l_0, m_0] = 1$ where $[x, y] = xyx^{-1}y^{-1}$, $m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}]$, and $l_1 = [m_1^{-1}, l_0][m_1^{-1}, l_0^{-1}]$. See [2].

THEOREM 2. *Elementary surgery along a doubled knot does not yield* $S^2 \times S^1$.

*Proof*. Perform a longitudinal surgery along $dK$ by replacing the regular neighborhood $f(T_1)$ of $dK$ by a solid torus $T_2$ to obtain $M^3 = \overline{S^3 - f(T_1)} \cup_h T_2$ where $h\colon \mathrm{Bd}\, T_2 \to \mathrm{Bd}\, f(T_1)$ is a homeomorphism which takes a meridian of $T_2$ to the longitude $f(l_1)$ of $f(T_1)$.

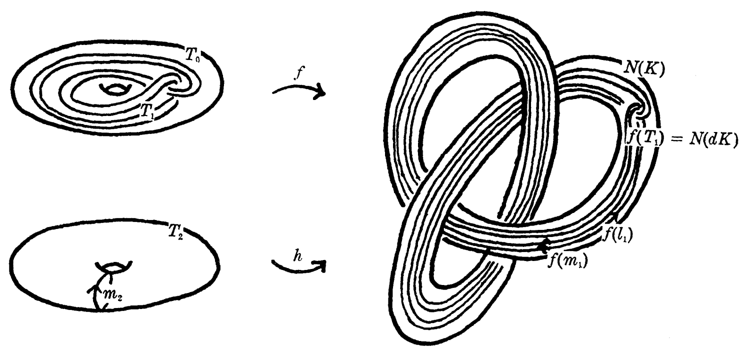


FIGURE 2.

Now instead of first replacing $N(K)$ by $T_0$ and then replacing $N(dK) = f(T_1)$ by $T_2$, first replace $T_1$ by $T_2$ and then replace $N(K)$ by $T_0$. Then by van Kampen's theorem, the fundamental group of $M^3$ is the free product with amalgamation $G(K)*G_2$ with the identification of subgroups $(l, m)$ of $G(K)$ and $(l_0, m_0)$ of $G_2$ where $G_2$ is obtained from $G_1$ by adding the relation $l_1 = 1$. The group $G_2$ has the following presentation: $G_2 = (l_0, m_1 \mid [l_0, m_0] = 1,\ \ m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}],\ \ l_1 = [m_1^{-1}, l_0][m_1^{-1}, l_0^{-1}] = 1)$. If we add the relation $m_1 l_0 = l_0^{-1} m_1$ to $G_2$, then $m_1^{-1} l_0 = l_0^{-1} m_1^{-1}$, and it follows that $m_0 = l_0^{-1} m_1 l_0 m_1^{-1} l_0^{-1} m_1^{-1} l_0 m_1 = l_0^{-4}$ and $l_1 = m_1^{-1} l_0 m_1 l_0^{-1} m_1^{-1} l_0^{-1} m_1 l_0 = 1$. Thus the relations $[l_0, m_0] = 1$ and $l_1 = 1$ are consequences of the relation $m_1 l_0 = l_0^{-1} m_1$, and the group $\bar{G}_2 = (\bar{l}_0, \bar{m}_1 \mid \bar{m}_1 \bar{l}_0 = \bar{l}_0^{-1} \bar{m}_1)$ is a quotient group of $G_2$. Now the properties of $\bar{G}_2$ are well-known: $\bar{G}_2$ is torsion-free and $\bar{l}_0 \neq 1$. Hence, $\bar{m}_0 = \bar{l}_0^{-4} \neq 1$ in $\bar{G}_2$, $m_0 \neq 1$ in $G_2$, and $m_0 \neq 1$ in $\pi_1(M^3)$. But $m_0 = [l_0^{-1}, m_1][l_0^{-1}, m_1^{-1}]$.

Thus $\pi_1(M^3)$ is not abelian, and $M^3$ is not homeomorphic to $S^2 \times S^1$. This completes the proof of Theorem 2.

Finally we consider composite knots. A knot $K$ is a composite of nontrivial knots $K_1$ and $K_2$ if there is a 2-sphere $S^2$ and an arc $\alpha$ in $S^2$ such that (1) $S^2 \cap K = \{x, y\}$ $(x \neq y)$ (2) $\alpha$ is an arc from $x$ to $y$ (3) $((\mathrm{Int}\, S^2) \cap K) \cup \alpha$ is a knot of the same type as $K_1$ (4) $((\mathrm{Ext}\, S^2) \cap K) \cup \alpha$ is a knot of the same type as $K_2$. The composite knot $K$ is denoted by $K_1 \# K_2$.

If $m_i$ is a meridian of $K_i$ and $l_i$ is a longitude of $K_i$ $(i = 1, 2)$, then the group of the composite knot, $G(K_1 \# K_2) = \pi_1(\overline{S^3 - N(K)})$, is the free product with amalgamation $G(K_1)*G(K_2)$ with the identification of subgroups $(m_1)$ of $G(K_1)$ and $(m_2)$ of $G(K_2)$ determined by $m_1 = m_2$. A longitude for $K_1 \# K_2$ is $l = l_1 l_2$. See [3]. By Theorem 1 it suffices to consider composite knots with trivial Alexander polynomial. Such a knot is the composite of two knots each with trivial Alexander polynomial. The following theorem will be proved, however, for arbitrary composite knots.

THEOREM 3. *Elementary surgery along a composite knot does not yield* $S^2 \times S^1$.

*Proof.* Perform a longitudinal surgery along $K_1 \# K_2$. The fundamental group of the surgered manifold $M^3$ is obtained by adding the relation $l = 1$ or $l_1 = l_2^{-1}$ to $G(K_1 \# K_2)$. Thus $\pi_1(M^3)$ can be considered as the free product with amalgamation $G(K_1)*G(K_2)$ with the identification of subgroups $(l_1, m_1)$ of $G(K_1)$ and $(l_2, m_2)$ of $G(K_2)$ determined by $l_1 = l_2^{-1}$ and $m_1 = m_2$. Since $K_i$ is nontrivial, $l_i \neq 1$ in $G(K_i)$, and so $l_i \neq 1$ in $\pi_1(M^3)$. But $l_i$ is in the commutator subgroup of $G(K_i)$, so also in the commutator subgroup of $\pi_1(M^3)$. Hence $\pi_1(M^3)$ is nonabelian, and $M^3$ is not homeomorphic to $S^2 \times S^1$. This establishes Theorem 3.

We conclude with the following conjecture: $S^2 \times S^1$ cannot be obtained by elementary surgery along any nontrivial knot. The proof of this conjecture like the proof of the conjecture, that elementary surgery along a nontrivial knot does not yield a counterexample to the Poincaré Conjecture, seems very difficult.

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