

ON PUISEUX SERIES
 WHOSE CURVES PASS THROUGH AN INFINITY
 OF ALGEBRAIC LATTICE POINTS

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1. Introduction. Runge [4] proved that certain binary Diophantine equations have only finitely many solutions. Here we give an argument concerning lattice points represented by Puiseux series which proves Runge's Theorem and permits a generalization which shows that there are only finitely many solutions in integers—subject to suitable restrictions—of an algebraic number field. As in the case of Runge's Theorem upper bounds for the absolute value of each solution can be computed, by the methods of the proof.

Let

$$F(x, y) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x^i y^j \in \mathbf{C}[x, y]$$

be of degree d_1 and d_2 in x and y , respectively. Let $\lambda > 0$. We define the λ -leading part, $F_\lambda(x, y)$, of $F(x, y)$ to be the polynomial consisting of the sum of all terms $a_{ij} x^i y^j$ of $F(x, y)$ for which $i + \lambda j$ is maximal, for that fixed value of λ . We define the leading part, $\tilde{F}(x, y)$, of $F(x, y)$ to be the sum of all such terms as λ varies.

We say that an irreducible polynomial

$$F(x, y) \in \mathbf{Z}[x, y]$$

satisfies *Runge's Condition* unless there exists a λ so that $\tilde{F} = F_\lambda$ is a constant multiple of a power of an irreducible polynomial.

Runge's Theorem can now be conveniently formulated: *If $F(x, y)$ satisfies Runge's Condition, then the Diophantine equation $F(x, y) = 0$ has only finitely many solutions $(x, y) \in \mathbf{Z}^2$.*

Let L denote an algebraic number field of degree t . Let the conjugates of $\theta \in L$ be denoted by $\theta^{(1)} = \theta, \theta^{(2)}, \theta^{(3)}, \dots, \theta^{(t)}$, and let

$$|\bar{\theta}| = \max_{1 \leq \tau \leq t} |\theta^{(\tau)}|.$$

Denote the ring of algebraic integers in L by \mathcal{O}_L . We say that $(x, y) \in \mathcal{O}_L^2$ is an L -lattice point. We consider certain analytic functions $y = f(x)$, of a complex

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variable x , represented by Puiseux series

$$f(x) = \sum_{n=-m}^{\infty} a_n x^{-n/e},$$

$m, e \in \mathbf{Z}$, $a_{-m} \neq 0$, $e > 0$, in a neighborhood of infinity, $|x| > R$.

THEOREM 1. *Let $y = f(x)$ where the Puiseux coefficients a_n , for $n \leq N$, lie in an algebraic number field K of degree s . Let M be the number of pairs of integers (μ, ν) with*

$$1 \leq \mu \leq \nu m/e, \quad 1 \leq \nu < se.$$

Consider L -lattice points (x, y) that satisfy

- (i) $y^{(\tau_0)} = f(x^{(\tau_0)})$, for some τ_0 with $|x^{\tau_0}| \geq |x|^{c_1}$,
- (ii) $|y| \leq c_2 |x|^{c_3}$,
- (iii) $|1/x| \leq c_4 |x|^{c_5}$,

where c_1, c_2, c_3, c_4 , and c_5 are positive constants with

$$c_5 < c_1/es(t-1).$$

Suppose that

$$\begin{aligned} N &\geq Me + m(se-1)/(c_1 - c_5 es(t-1)), \\ N &\geq c_3 e/c_5, \quad N \geq m/c_5. \end{aligned}$$

Then all these L -lattice points (x, y) satisfy an equation $P(x, y) = 0$ where $P(x, y) \in \mathbf{Z}[x, y]$, $\deg_y P \leq se$, and P_λ is a monomial for all $\lambda \neq m/e$. For $\lambda = m/e$, P_λ is a constant multiple of a power of x times a power of an irreducible polynomial in $\mathbf{Z}[x, y]$. Moreover, $\tilde{P} = P_{m/e}$. If there is an infinity of L -lattice points satisfying (i), (ii) and (iii), then $P(x, f(x)) \equiv 0$.

We deduce, in [3], Runge's Theorem, as a corollary of Theorem 1, in the case $t = 1$. Bounds for $|x|$ and $|y|$ are also computed in [3].

For general t we deduce, in [3], the following generalization of Runge's Theorem, as another corollary of Theorem 1.

THEOREM 2. *Let $F(x, y)$ be irreducible in $\mathbf{Z}[x, y]$ and set*

$$F(x, y) = \sum_{i=1}^{d_1} x^i f_i(y) = \sum_{j=0}^{d_2} y^j g_j(x)$$

where

$$g_{d_2}(x) = a \prod_{r=1}^v (x - \alpha_r)^{\mu_r}, \quad f_{d_1}(y) = b \prod_{s=1}^w (y - \beta_s)^{\nu_s}$$

and assume that there exist infinitely many L -lattice points $(x, y) \in \mathcal{O}_L^2$ which satisfy $F(x, y) = 0$ and

$$(iv) \quad |1/x| \leq c_6 |x|^{1/(t-1)d_2^2}, \quad |1/y| \leq c_6 |y|^{1/(t-1)d_1^2},$$

where c_6 is a positive constant. Assume one of the following conditions for these L -lattice points:

$$(v) \quad \begin{aligned} |1/(x - \alpha_r)| &\leq c_7 |y|^{1/\mu_r - c_8}, & r = 1, 2, \dots, v, \\ |1/(y - \beta_s)| &\leq c_7 |x|^{1/\nu_s - c_8}, & s = 1, 2, \dots, w, \end{aligned}$$

where c_7, c_8 are positive constants. Then $F(x, y)$ violates Runge's Condition.

To see that Theorem 2 is a corollary of Theorem 1, we need to observe that $e \leq d_2$, with strict inequality if Runge's Condition holds.

2. Outline of the proof of Theorem 1. Let $\rho = 1, 2, \dots, M$ and construct functions

$$F(x, y; \rho) = s^\rho \prod_{\sigma=1}^s \prod_{\epsilon=0}^{\epsilon-1} \left(y - \sum_{n=-m}^N a_n^{(\sigma)} \zeta^\epsilon x^{-1/\epsilon} \right)^n.$$

Estimate the quantities $F(x^{(\tau)}, y^{(\tau)}; \rho)$ to find, for $\tau = \tau_0$, that they are of the order of magnitude $O(|x|^k)$ where $k \leq M + (se - 1)m/e - c_1(N + 1)/e$; whereas for a general τ they are $O(|x|^{cs^{sN}})$. Observe that

$$F(x, y; \rho) = P(x, y; \rho) + \sum_{\mu} \sum_{\nu} b_{\rho\mu\nu} y^\nu / x^\mu + O(x^{-1/\epsilon}),$$

where $P(x, y; \rho)$ is a polynomial with rational coefficients, the $b_{\rho\mu\nu}$'s are rational numbers, and where the double sum is extended over the M values of (μ, ν) indicated in the theorem. Construct integers B_ρ , not all zero, for which

$$\sum_{\rho=0}^M B_\rho b_{\rho\mu\nu} = 0,$$

for all M pairs (μ, ν) , and for which

$$Q(x, y) = \sum_{\rho=0}^M B_\rho P(x, y; \rho) \in \mathbf{Z}[x, y].$$

From the estimations of $F(x^{(\tau)}, y^{(\tau)}; \rho)$ we deduce that $Q(x, y)$ vanishes for all but a finite number of these L -lattice points (x, y) , and so there is a polynomial of the form

$$P(x, y) = Q_1(x)Q(x, y) \in \mathbf{Z}[x, y],$$

with $Q_1(x) \in \mathbf{Z}[x]$, that vanishes at all of these L -lattice points. Finally we argue that $\tilde{P}(x, y)$ and $P_\lambda(x, y)$ have the indicated properties.

Note that for the finiteness results of Theorems 1 and 2 over algebraic number fields some restrictions like (iv) and (v) in Theorem 2 are needed. For example the equation $(x - a)(y - b) = 1$ satisfies Runge's Condition, but has infinitely many solutions, $x = a + \eta$, $y = b + \eta^{-1}$ in integers of a number field with infinitely many units η .

As a simple illustration we consider the problem: When is the product of four consecutive integers a square? That is, find the solutions of

$$(2.1) \quad \begin{aligned} y^2 &= (x - 1)x(x + 1)(x + 2) = x^4 + 2x^3 - x^2 - 2x \\ &= (x^2 + x - 1)^2 - 1. \end{aligned}$$

This equation satisfies Runge's Condition. In \mathbf{Z} the only solutions are clearly $y = 0$, $x = -2, -1, 0, 1$. It is equally easy to see that there are no other solutions in integers of imaginary quadratic fields, where the only new units to be considered are $\omega = \pm i$, $(\pm 1 \pm i\sqrt{3})/2$. Now equation (2.1) yields

$$y + (x^2 + x - 1) = \omega, \quad y - (x^2 + x - 1) = -\bar{\omega}$$

and hence

$$y = \frac{1}{2}(\omega - \bar{\omega}), \quad x^2 + x - 1 = \frac{1}{2}(\omega + \bar{\omega}).$$

For $\omega = \pm i$ the equation $x^2 + x - 1 = 0$ has no solution in the field and for $\omega = (\pm 1 \pm i\sqrt{3})/2$ the quantities $(\omega \pm \bar{\omega})/2$ are not integers.

In some real quadratic number fields there obviously are solutions with $y \neq 0$. Nevertheless the conditions of Theorem 2 are satisfied and we can get good bounds for the solutions. To see this let η be the fundamental unit of the field. Then (2.1) yields

$$y + (x^2 + x - 1) = \pm \eta^n, \quad y - (x^2 + x - 1) = \mp \eta^{-n},$$

or

$$(2.2) \quad y = \pm \frac{1}{2}(\eta^n - \eta^{-n}), \quad x^2 + x - 1 = \pm \frac{1}{2}(\eta^n + \eta^{-n}).$$

So, if there are solutions for large n , then

$$(2.3) \quad |y| \sim \eta^n/2, \quad |x| \sim (\eta^n/2)^{1/2}$$

while

$$|1/y| \sim 2/\eta^n, \quad |1/x| \sim (\eta^n/2)^{-1/2}.$$

Thus condition (iv) of Theorem 2 is satisfied and condition (v) is vacuous. This yields the fact that the number of solutions is finite in integers of any real quadratic number field and the bounds can be readily computed, as we do in [3] for solutions in \mathbf{Z} .

The above argument is equally valid for all number fields with a cyclic group of units (other than the roots of unity), since in those cases $|1/x|$ and $|1/y|$ are small when n —and hence $|x|$ and $|y|$ —are large.

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