ON WAVE MOTION IN AN INFINITE SOLID BOUNDED INTERNALLY BY A CYLINDER OR A SPHERE

ARNOLD N. LOWAN

PART I

In two previous papers,† the author investigated the problem of wave motion for infinite domains of one, two, and three dimensions and for certain sub-infinite domains; that is, domains bounded in certain directions but extending to infinity in other directions. The present paper is a sequel to the aforementioned papers and deals with the problem of wave motion in an infinite solid, bounded internally by a cylinder or a sphere.

In the subsequent developments we shall use the following abbreviations:

$$\sigma(\alpha) = (a^2\alpha^2 - k^2)^{1/2}, \qquad s(p, \alpha) = \alpha^2 + (p^2 - k^2)/a^2,$$

where α is a real variable ranging from $-\infty$ to ∞ and p is a complex variable whose real part is positive. We shall also introduce the operators ∇_c, ∇_s, $\sum\int\int\int$, and $\sum\int\int\int\int$ defined as follows:

$$\nabla_c = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{a^2}(p^2 - k^2) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

$$\nabla_s = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{1}{a^2}(p^2 - k^2),$$

$$\sum\int\int\int \{F_n(r', \theta', \alpha)\} = \sum_{n=0}^{\infty}(2n+1)\cos n(\theta - \theta')\int_R^\infty r'dr' \cdot \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} \alpha F_n(r', \theta', \alpha)d\alpha,$$

$$\sum\int\int\int\int \{F_n(r', \theta', \phi', \alpha) = \sum_{n=0}^{\infty}(2n+1)P_n(\cos\gamma) \cdot \int_R^\infty r'^{3/2}dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} \alpha F_n(r', \theta', \phi', \alpha)d\alpha,$$

† *On wave motion for infinite domains*, Philosophical Magazine, (7), vol. 26 (1938), pp. 340–360; *On wave motion for sub-infinite domains*, Philosophical Magazine, (7), vol. 27 (1939), pp. 182–194. These papers will be referred to as L-1 and L-2, respectively.

where P_n is the Legendre polynomial of degree n and γ is the angle between the vectors from the origin to the points (r, θ, ϕ) and (r', θ', ϕ').

Consider first the case of an infinite solid bounded internally by a cylinder. In this case the displacement satisfies the system A consisting of equations (1), (2), (3), and (4):

$$\nabla^2 U(P; t) = 2b \frac{\partial}{\partial t} U(P; t) + \frac{1}{a^2} \frac{\partial^2}{\partial t^2} U(P; t) + \Phi(P; t), \tag{1}$$

$$\lim_{t \to 0} U(P; t) = f(P), \tag{2}$$

$$\lim_{t \to 0} \frac{\partial}{\partial t} U(P; t) = g(P), \tag{3}$$

$$U(P; t) = \phi(\theta; t), \qquad r = R. \tag{4}$$

As in L-1 and L-2, we put

$$U(P; t) = u(P; t) + v(P; t), \tag{5}$$

where $u(P; t)$ satisfies the system B, consisting of (1), (2), (3), and the boundary condition

$$u(P; t) = 0, \qquad r = R, \tag{6}$$

and where $v(P; t)$ satisfies the system C, obtained from A, by replacing $U(P; t)$ by $v(P; t)$ and putting $f(P) = g(P) = \Phi(P; t) = 0$. We proceed to the solution of the systems B and C.

As in L-1 and L-2, the method of solving the systems B and C consists in making the substitution

$$u(r, \theta; t) = e^{-kt} u_1(r, \theta; t), \tag{7}$$

$$\Phi(r, \theta; t) = e^{-kt} \Phi_1(r, \theta; t), \tag{8}$$

$$v(r, \theta; t) = e^{-kt} v_1(r, \theta; t), \tag{9}$$

where $k = ba^2$.

Let B_1 and C_1 designate the systems obtained from the systems B and C, by the substitutions (7), (8), and (9) for the functions u_1, v_1, and Φ_1. The solutions of the last two systems are obtained by operating on systems B and C with the Laplace operator and obtaining the systems B_1^* and C_1^*, for the Laplace transforms $u_1^*(r, \theta; p)$ and $v_1^*(r, \theta; p)$. When the solutions of B_1^* and C_1^* have been obtained, the corresponding solutions of B and C are obtained by acting on the corresponding solutions with the inverse Laplace operator. The system B_1^* is ultimately obtained in the following form:

$$\nabla_c u_1^*(r, \theta; p) = -\frac{1}{a^2}\{pf(r, \theta) + h(r, \theta)\} + \Phi_1^*(r, \theta; p)$$
$$(10) \qquad = -F(r, \theta; p) \text{ (say)},$$
$$h(r, \theta) = g(r, \theta) + kf(r, \theta),$$
$$(11) \qquad u_1^*(R, \theta; p) = 0,$$

and the system C_1^* in the form

$$(12) \qquad \nabla_c v_1^*(r, \theta; p) = 0,$$
$$(13) \qquad v_1^*(R, \theta; p) = \phi_1^*(\theta; p).$$

In order to obtain the solution of B_1^*, we make use of the identity†

$$(14) \qquad f(r, \theta) = \frac{1}{4\pi}\sum\int\int\int \{f(r', \theta')G_n(r, r'; \alpha)\}$$

where

$$G_n(r, r'; \alpha) = \frac{H_n^1(\alpha r)}{H_n^1(\alpha R)}\{J_n(\alpha r')\cdot H_n^1(\alpha R) - J_n(\alpha R)H_n^1(\alpha r')\}$$

for $r' < r$; the corresponding expression for $r' > r$, is obtained by interchanging r and r'.

In view of (14), it can be verified that the expression

$$(15) \qquad u_1^*(r, \theta; p) = \frac{1}{4\pi}\sum\int\int\int \left\{F(r', \theta'; p)\cdot\frac{G_n(r, r'; \alpha)}{s(p, \alpha)}\right\}$$

is the solution of the system B_1^*.

Bearing in mind the significance of $F(r, \theta; p)$ from (10), (15) becomes

$$(16) \qquad u_1^*(r, \theta; p) = u_{1,1}^*(r, \theta; p) + u_{1,2}^*(r, \theta; p) + u_{1,3}^*(r, \theta; p),$$

where

$$(17) \qquad u_{1,1}^*(r, \theta; p) = \frac{1}{4\pi a^2}\sum\int\int\int \left\{f(r', \theta')\frac{pG_n(r, r'; \alpha)}{s(p, \alpha)}\right\},$$

$$(18) \qquad u_{1,2}^*(r, \theta; p) = \frac{1}{4\pi a^2}\sum\int\int\int \left\{h(r', \theta')\frac{G_n(r, r'; \alpha)}{s(p, \alpha)}\right\},$$

$$(19) \qquad u_{1,3}^*(r, \theta; p) = -\frac{1}{4\pi}\sum\int\int\int \left\{G_n(r, r'; \alpha)\frac{\Phi_1^*(r', \theta'; p)}{s(p, \alpha)}\right\}.$$

† See Appendix at end of this paper, §1.

The problem now reduces to obtaining the inverse Laplace transforms of (17), (18), and (19). Making use of the identities

$$\int_0^\infty e^{-pt} \cos \lambda t \, dt = \frac{p}{p^2 + \lambda^2}, \quad \int_0^\infty e^{-pt} \sin \lambda t \, dt = \frac{\lambda}{p^2 + \lambda^2},$$

and of Borel's theorem, we finally get

$$\text{(20)} \quad u_{1,1}(r, \theta; t) = \frac{1}{4\pi} \sum \iiint \{ f(r', \theta') G_n(r, r'; \alpha) \cos \sigma(\alpha) t \},$$

$$\text{(21)} \quad u_{1,2}(r, \theta; t) = \frac{1}{4\pi} \sum \iiint \left\{ h(r', \theta') G_n(r, r'; \alpha) \frac{\sin \sigma(\alpha) t}{\sigma(\alpha)} \right\},$$

$$\text{(22)} \quad \begin{aligned} &u_{1,3}(r, \theta; t) \\ &= -\frac{a^2}{4\pi} \int_0^t \sum \iiint \left\{ \Phi_1(r', \theta'; t - \tau) G_n(r, r'; \alpha) \frac{\sin \sigma(\alpha) \tau}{\sigma(\alpha)} \right\} d\tau. \end{aligned}$$

In the case where the solutions f, g, ϕ, and Φ do not depend on θ, it is clear that the final solution is also independent of θ. In this case, it can be shown (see Appendix, §2) that the identity (14) must be replaced by

$$\text{(14')} \qquad f(r) = \frac{1}{4\pi} \int_R^\infty r' dr' \int_{-\infty}^\infty \alpha f(r') G_0(r, r'; \alpha) d\alpha,$$

where the expression for G_0 is obtained from that for G_n, by replacing the index n by zero.

In view of this result, the solutions for $u_{1,1}$, $u_{1,2}$, and $u_{1,3}$, when these solutions do not depend on θ, may be obtained at once from (20), (21), and (22), by dropping the summation sign and the factor $\cos n(\theta - \theta')$ and replacing the subscript n by zero.

We proceed to the solution of the system C_1^*. The expression

$$\text{(23)} \quad v_1^*(r, \theta; p) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \phi_1^*(\theta'; p) \cos n(\theta - \theta') w_n^*(r; p) d\theta',$$

where

$$\text{(24)} \qquad w_n^*(r; p) = \frac{H_n^1(\alpha r)}{H_n^1(\alpha R)} \text{ (say)},$$

is a solution of (12), satisfying the boundary conditions (13), provided

$$\text{(25)} \qquad s(p, \alpha) = 0.$$

If the function $w_n(r, t)$ is defined by

$$\int_0^\infty e^{-pt} w_n(r;\, t)dt = \frac{H_n^1(\alpha r)}{H_n^1(\alpha R)} = w_n^*(r;\, p) \text{ (say)}, \tag{26}$$

then by Borel's theorem, the inverse Laplace transform of (23) is

$$v_1(r, \theta;\, t) = \frac{1}{\pi} \sum_{-\infty}^{\infty} \int_0^{2\pi} \int_0^t \phi_1(\theta';\, t-\tau) w_n(r, \tau) \cos n(\theta - \theta') d\theta' d\tau. \tag{27}$$

Our problem thus reduces to solving the integral equation (26), where α is defined in (25). Since $H_n^1(z) \rightarrow 0$ in the upper half of the z-plane and since its zeros are known to lie in the lower half of the plane, it can be easily shown with the aid of the Cauchy integral theorem that

$$\begin{aligned} w_n^*(r;\, p) &= \frac{H_n^1(r\alpha)}{H_n^1(R\alpha)} = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{x}{x^2 - \alpha^2} \cdot \frac{H_n^1(rx)}{H_n^1(Rx)} dx \\ &= \frac{a^2}{\pi i} \int_{-\infty}^{\infty} \frac{x}{p^2 + [\sigma(x)]^2} \cdot \frac{H_n^1(rx)}{H_n^1(Rx)} dx. \end{aligned} \tag{28}$$

The inversion of (26) leads at once to

$$w_n(r;\, t) = \frac{a^2}{\pi i} \int_{-\infty}^{\infty} \frac{x \sin \sigma(x) t}{\sigma(x)} \frac{H_n^1(rx)}{H_n^1(Rx)} dx. \tag{29}$$

In (27) and (29), we have the complete solution of the system B_1.

It should be remarked that the expression $w_n(r;\, t)$ given by (29) is real. Indeed, if in the contribution to the integral in (29) for the range from $-\infty$ to 0 we make the substitution $x = -\xi$, replace once more the variable of integration ξ by x, and furthermore, make use of the well known relation

$$H_n^1(-z) = (-1)^n [H_n^1(z) - 2J_n(z)], \tag{30}$$

(29) becomes

$$\begin{aligned} w_n(r;\, t) = \frac{a^2}{\pi i} \int_0^\infty \frac{x \sin \sigma(x) t}{\sigma(x)} \bigg\{ & \frac{H_n^1(rx)}{H_n^1(Rx)} \\ & - \frac{H_n^1(rx) - 2J_n(rx)}{H_n^1(Rx) - 2J_n(Rx)} \bigg\} dx. \end{aligned} \tag{31}$$

Since $H_n^1(z) = J_n(z) + iY_n(z)$, the above equation ultimately becomes

$$w_n(r;\, t) = \frac{2a^2}{\pi} \int_0^\infty \frac{x \sin \sigma(x) t}{\sigma(x)} \cdot \frac{J_n(Rx) Y_n(rx) - J_n(rx) Y_n(Rx)}{(J_n(Rx))^2 + (Y_n(Rx))^2} dx. \tag{32}$$

If the function ϕ is independent of θ, it is clear that v_1 is independent of θ. In this case, instead of (23), we start with

$$v_1^*(r;\, p) = \phi_1^*(p) w_0^*(r;\, p), \tag{23'}$$

where $w_0^*(r;\, p)$ may be obtained from (28) by replacing the subscript n by zero. The corresponding expression for $w_0(r,\, t)$ may therefore be obtained from (29) by replacing the subscript n by zero. With this definition of $w_0(r;\, t)$ we have

$$v_1(r;\, t) = \int_0^\infty \phi_1(t - \tau) w_0(r;\, \tau) d\tau.$$

Part II. Case of an infinite solid bounded internally by a sphere

In this case, the displacement $U(P;\, t)$ must satisfy a system similar to A, except that $\nabla^2 u(P;\, t)$ is now the Laplacian in spherical coordinates and the boundary condition (4) may assume the more general form

$$U(P;\, t) = \Omega(\theta,\, \phi;\, t), \qquad r = R. \tag{4'}$$

The method of solution is entirely similar to that in Part I. The Laplace transforms of the functions $u_1(P;\, t)$ and $v_1(P;\, t)$ obtained from $u(P;\, t), v(P;\, t)$ by the substitutions (7), (8), and (9), for the case under consideration, must now satisfy the systems D_1^*, consisting of equations (33) and (34), and E_1^*, consisting of (35) and (36):

$$\begin{aligned} \nabla_s u_1^*(r,\, \theta,\, \phi;\, p) &= -\frac{1}{a^2}\{pf(r,\, \theta,\, \phi) + h(r,\, \theta,\, \phi)\} + \Phi_1^*(r,\, \theta,\, \phi;\, p) \\ &= -F(r,\, \theta,\, \phi;\, p) \text{ (say)}, \end{aligned} \tag{33}$$

$$u_1^*(R,\, \theta,\, \phi;\, p) = 0, \tag{34}$$

$$\nabla_s v_1^*(r,\, \theta,\, \phi;\, p) = 0, \tag{35}$$

$$v_1^*(r,\, \theta,\, \phi;\, p) = \Omega_1^*(\theta,\, \phi;\, p). \tag{36}$$

We proceed to the solution of the system D_1^*. In this case, we make use of the following identity;†

$$f(r,\, \theta,\, \phi) = \frac{1}{8\pi r^{1/2}} \sum \int\!\!\int\!\!\int\!\!\int \{f(r',\, \theta',\, \phi') G_{n+1/2}(r,\, r',\, \alpha)\}. \tag{37}$$

As in Part I, it can be verified that the expression

† The derivation of this identity is discussed in the Appendix, §4. The expression for $G_{n+1/2}(r,\, r',\, \alpha)$ may be obtained from that of $G_n(r,\, r',\, \alpha)$ by replacing the subscript n by $n+1/2$.

$$(38)\quad u_1^*(r, \theta, \phi; p) = \frac{1}{8\pi r^{1/2}} \sum \int\int\int\int \left\{ F(r', \theta', \phi') \frac{G_{n+1/2}(r, r', \alpha)}{s(p, \alpha)} \right\}$$

is a solution of the system D_1^*, provided that the expression for $G_{n+1/2}(r, r', \alpha)$ in (38) is obtained from the expression $G_n(r, r', \alpha)$ by replacing the subscript n by $n+1/2$. The expressions for $u_{1,1}$, $u_{1,2}$, and $u_{1,3}$ may therefore be obtained at once from the corresponding expressions in Part I in the forms

$$(39)\quad \begin{aligned} u_{1,1}(r, \theta, \phi; t) = \frac{1}{8\pi r^{1/2}} \sum \int\int\int\int \{ & f(r', \theta', \phi') \\ & \cdot G_{n+1/2}(r, r'; \alpha) \cos \sigma(\alpha) t \}, \end{aligned}$$

$$(40)\quad \begin{aligned} u_{1,2}(r, \theta, \phi; t) = \frac{1}{8\pi r^{1/2}} \sum \int\int\int\int \Big\{ & h(r', \theta', \phi') \\ & \cdot G_{n+1/2}(r, r'; \alpha) \frac{\sin \sigma(\alpha) t}{\sigma(\alpha)} \Big\}, \end{aligned}$$

$$(41)\quad \begin{aligned} u_{1,3}(r, \theta, \phi; t) = -\frac{a^2}{8\pi r^{1/2}} \int_0^t \sum \int\int\int\int \Big\{ & \Phi_1(r', \theta', \phi'; t - \tau) \\ & \cdot G_{n+1/2}(r, r'; \alpha) \frac{\sin \sigma(\alpha) \tau}{\sigma(\alpha)} \Big\} d\tau. \end{aligned}$$

We now proceed to the solution of the system E_1^*. The expression

$$(42)\quad \begin{aligned} v_1^*(r, \theta, \phi; p) = \sum_{n=0}^{\infty} \int_0^{\pi} \int_0^{2\pi} & (2n+1) P_n(\cos \gamma) \\ & \cdot \frac{H^1_{n+1/2}(\alpha r)}{H^1_{n+1/2}(\alpha R)} \Omega_1^*(\theta', \phi'; p) \sin \theta' d\theta' d\phi' \end{aligned}$$

is a solution of E_1^*, provided

$$(43)\quad s(p, \alpha) = 0$$

(see Appendix, §3).

If, then, the function $w_{n+1/2}(r, t)$ is defined by

$$(44)\quad \int_0^{\infty} e^{-pt} w_{n+1/2}(r; t) dt = \frac{H^1_{n+1/2}(\alpha r)}{H^1_{n+1/2}(\alpha R)} = w^*_{n+1/2}(r; p) \text{ (say)},$$

then by Borel's theorem, the inversion of (42) yields

$$(45)\quad \begin{aligned} v_1(r, \theta, \phi; t) = \sum_{n=0}^{\infty} \int_0^{\pi} \int_0^{2\pi} \int_0^t & (2n+1) P_n(\cos \gamma) \\ & \cdot \Omega_1(\theta', \phi'; t - \tau) w_{n+1/2}(r', \tau) d\theta' d\phi' d\tau. \end{aligned}$$

Our problem thus reduces to solving the integral equation (44).

This equation is identical with (26) except that the subscript n is replaced by $n+1/2$. The method of solution of (45) is entirely similar to that of (26), and we finally get

$$w_{n+1/2}(r;\, t) = \frac{a^2}{\pi i}\int_{-\infty}^{\infty} \frac{x \sin \sigma(x)t}{\sigma(x)} \cdot \frac{H^1_{n+1/2}(rx)}{H^1_{n+1/2}(Rx)}\, dx, \tag{46}$$

or (by a transformation similar to that for (29))

$$\begin{aligned} w_{n+1/2}(r;\, t) = \frac{2a^2}{\pi}\int_0^{\infty} &\frac{x \sin \sigma(x)t}{\sigma(x)} \\ &\cdot \frac{J_{n+1/2}(Rx)Y_{n+1/2}(rx) - J_{n+1/2}(rx)Y_{n+1/2}(Rx)}{(J_{n+1/2}(Rx))^2 + (Y_{n+1/2}(Rx))^2}\, dx. \end{aligned} \tag{47}$$

The complete solution in the case of an infinite solid bounded internally by a sphere is therefore given by

$$U(P;\, t) = e^{-ba^2 t}(u_{1,1} + u_{1,2} + u_{1,3} + v), \tag{48}$$

where the terms in parentheses are given by (39), (40), (41), and (45).

An important special case is that in which the functions f, g, Ω, Φ do not depend on the angles θ and ϕ. While the solution can be obtained from the previous solution by integrating the variables θ' and ϕ', it is easier to proceed as follows. We have to solve the system of equations

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = 2b\frac{\partial u}{\partial t} + \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + \Phi(r;\, t), \qquad R < r < \infty, \tag{49}$$

$$\lim_{t\to 0} u(r;t) = f(r), \qquad R < r < \infty, \tag{50}$$

$$\lim_{t\to 0} \frac{\partial}{\partial t} u(r;\, t) = g(r), \qquad R < r < \infty, \tag{51}$$

and

$$U(R;\, t) = \phi(t).$$

If we make the substitution $u(r;\, t) = (1/r)v(r;\, t)$, it is readily seen that the function $v(r;\, t)$ must satisfy the system

$$\frac{\partial^2 v}{\partial r^2} = 2b\frac{\partial v}{\partial t} + \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + r\Phi(r;\, t), \tag{52}$$

$$\lim_{t\to 0} v(r;\, t) = rf(r), \tag{53}$$

$$\lim_{t\to 0} \frac{\partial}{\partial t} v(r;\, t) = rg(r), \tag{54}$$

$$V(R;\, t) = R\phi(t). \tag{55}$$

The system satisfied by $v(r,\, t)$ is formally identical with that corresponding to wave motion in a semi-infinite solid extending from $r = R$ to $r = \infty$.

The solution corresponding to a solid extending from 0 to ∞ is given in L-2. It is clear that our present solution may be obtained by replacing x by $r - R$, in L-2.

Appendix

1. **Derivation of the identity** (14). Consider the problem of heat conduction in an infinite solid, bounded internally by a cylinder, the surface of which is kept at 0°. The solution of this problem may be obtained with the aid of the appropriate Green's function for a point source. The expression for the Green's function is

$$\text{(I)}\quad \begin{aligned} G(r, \theta; t; r', \theta') = \frac{1}{4\pi} \sum_{-\infty}^{\infty} \cos n(\theta - \theta') \int_{-\infty}^{\infty} \alpha e^{-k\alpha^2 t} \\ \cdot \frac{H_n^1(\alpha r')}{H_n^1(\alpha R)} \left\{ J_n(\alpha r) H_n^1(\alpha R) - J_n(\alpha R) H_n^1(\alpha r) \right\} d\alpha \end{aligned}$$

for $r < r'$.

In the case $r > r'$, the corresponding expression may be obtained by interchanging r and r'. With the aid of the general formulas of Carslaw's† article 80, the solution of the problem of heat conduction under consideration is seen to be

$$\text{(II)}\quad T(r, \theta; t) = \frac{1}{4\pi} \sum \int\int\int \left\{ f(r', \theta') P_n(\cos \gamma) \cdot e^{-k\alpha^2 t} G_n(r, r', \alpha) \right\}.$$

Putting $t = 0$, we obtain the identity (14).

2. **Derivation of identity** (14′). The expression for the Green's function corresponding to a cylindrical source may be obtained by considering a continuous distribution of line sources around the cylinder $r = r'$, integrating for the variable θ' and dividing by 2π.

The corresponding solution of the problem in heat conduction can then be obtained from (II) by dropping the summation sign and the factor $\cos n(\theta - \theta')$ and replacing the subscript n by zero. If in this final solution we make $t = 0$, we obtain the identity (14′).

† Carslaw, *Mathematical Theory of Heat Conduction.*

3. **Derivation of identity** (42). Starting with the well known expansion

$$F(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_0^{\pi} \int_0^{2\pi} \frac{(n - |m|)!}{(n + |m|)!} \tag{56}$$
$$\cdot F(\theta', \phi') P_n^m(\cos \theta) P_n^m(\cos \theta') e^{im}(\phi - \phi') \sin \theta' d\theta' d\phi',$$

where the P_n^m's are the associated Legendre polynomials, and making use of the identity (see Carslaw's article 93)

$$P_n(\cos \gamma) = P_n(\cos \theta) P_n(\cos \theta') + 2 \sum_{m=1}^{\infty} \frac{(n - |m|)!}{(n + |m|)!} P_n^m(\cos \theta) \tag{57}$$
$$\cdot P_n^m(\cos \theta') \cos m(\phi - \phi'),$$

we find that the second member of (42) reduces to $\Omega_1^*(\theta, \phi; p)$ for $r = R$.

4. **Derivation of identity** (37). The Green's function corresponding to a point source in an infinite solid bounded internally by a sphere, the surface of which is kept at 0°, is given by

$$G(r, \theta, \phi; t; r_0, \theta_0, \phi_0) = \frac{1}{8\pi (r r_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) \tag{58}$$
$$\cdot \int_{-\infty}^{\infty} \alpha e^{-k\alpha^2 t} \frac{H_{n+1/2}^1(\alpha r_0)}{H_{n+1/2}^1(\alpha R)}$$
$$\cdot \left\{ J_n(\alpha r) H_{n+1/2}^1(\alpha R) - J_{n+1/2}(\alpha R) H_{n+1/2}^1(\alpha r) \right\} d\alpha.$$

With the aid of the general formula of Carslaw's article 80, the solution of the appropriate problem in heat conduction is found to be

$$T(r, \theta, \phi; t) = \frac{1}{8\pi r^{1/2}} \sum \int\int\int\int \left\{ e^{-k\alpha^2 t} f(r', \theta', \phi') G_{n+1/2}(r, r') \right\}. \tag{59}$$

Putting $t = 0$, we obtain identity (37).

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