Multiple Periodic solutions and Positive Homoclinic Solution for a differential equation

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Abstract

We consider the nonautonomous differential equation of second order $x'' - a(t)x + b(t)x^2 + c(t)x^3 = 0$, where $a(t), b(t), c(t)$ are T-periodic functions. This is a biomathematical model of an aneurysm in the circle of Willis. We prove the existence of at least two T-periodic solution for this equation, using coincidence degree theories.

1 Introduction

We consider the nonautonomous differential equation

$$x'' - a(t)x + b(t)x^2 + c(t)x^3 = 0, \tag{1.1}$$

where $a(t), b(t), c(t)$ are continuous T-periodic functions. The equation (1.1) comes from biomathematics model and was suggested by J. Cronin in [2] and G. Austin in [1].

The existence of periodic solutions to (1.1) was previously considered in [3], where $a(t), b(t), c(t)$ are T-periodic functions, subject to the constraints

$$0 < a \leq a(t) \leq A, |b(t)| \leq B, 0 < c \leq c(t) \leq C,$$

where $a, A, B, c, C > 0$ are constants, using basically a "Mountain Pass Lemma". For more details on these methods, see E. A. B. Silva [10].

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It follows from [3] that there exist periodic solutions, but the multiplicity of T-periodic solutions is not proved there. The applied method was a variational one, using a critical point theorem in finite dimensional spaces due to P. Rabinowitz [8], Galerkin type procedure and Gagliardo-Nirenberg inequalities.

The existence of a nontrivial positive homoclinic solution for equation (1.1) follows from [5], when $a(t), b(t), c(t)$ are 2π-periodic continuous functions, subject to the constraints

$$0 < a \leq a(t), 0 \leq b \leq b(t) \leq B, 0 < c \leq c(t) \leq C \tag{1.2}$$

and

$$B^2 - b^2 < 4ac, \tag{1.3}$$

where $a, b, A, B, c, C > 0$ are constants.

In this paper we prove the existence of at least two nontrivial periodic solutions with the same period T, to the equation (1.1) with $a(t), b(t), c(t)$ continuous T-periodic functions, using coincidence degree theory, under some specific assumptions to be given latter.

We also discuss the existence of positive homoclinic solutions to the equation (1.1), where $a(t), b(t), c(t)$ are 2π-periodic continuous functions, subject only to the constraints

$$0 < a \leq a(t), 0 \leq b \leq b(t) \leq B, 0 < c \leq c(t) \leq C, \tag{1.4}$$

where $a, b, A, B, c, C > 0$ are constants. We prove this result by changing some arguments in the proof of Grossinho, Minhóz and Tersian [5, Proposition 2 and Theorem 3].

2 Description of the Main Result

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. In order to make this presentation as self-contained as possible we introduce a few concepts and results about the coincidence degree as follows. For more details see R. E. Gaines and J. L. Mawhin [7].

Definition 2.1. Let X, Y be real Banach spaces, $L : DomL \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L is said to be a Fredholm mapping of index zero, if

$$\dim KerL = codimImL < +\infty$$

and ImL is closed in Y.

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$, such that

$$ImP = KerL$$

and
$$KerQ = ImL = Im(I - Q).$$
It follows that the restriction L_P of L to $DomL \cap KerP : (I - P)X \to ImL$ is invertible. Denote the inverse of L_P by K_P.

Definition 2.2. The mapping N is said to be L-compact on $\overline{\Omega}$, if Ω is an open bounded subset of X, $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Since ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \to KerL$.

We shall be interested in proving the existence of solutions for the operator equation
$$Lx = Nx, \tag{2.1}$$
a solution being an element of $DomL \cap \overline{\Omega}$ verifying (2.1).

The following results is due R. E. Gaines and J. L. Mawhin [7].

Proposition 2.3 (Mawhin's Continuation Theorem)**.** *Let L be a Fredholm mapping of index* 0 *and let N be L-compact on* $\bar{\Omega}$. *Suppose*

(1) *For each* $\lambda \in (0,1)$, $x \in \partial\Omega$
$$Lx \neq \lambda Nx.$$

(2) $QNx \neq 0$ *for each* $x \in KerL \cap \partial\Omega$ *and*
$$deg(JQN, \Omega \cap KerL, 0) \neq 0,$$
where $J : ImQ \to KerL$ *is an isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution in* $DomL \cap \bar{\Omega}$.

Our main theorem is

Theorem 2.4. *Consider* $0 < T$ *and let* $a(t), b(t), c(t)$ *be continuos T-periodic functions with*
$$0 < a \le a(t) \le A, 0 \le b \le b(t) \le B, 0 < c \le c(t) \le C, \tag{2.2}$$
$$B^2 - b^2 < 2ac, \tag{2.3}$$
where $a, b, A, B, c, C > 0$ *are constants and* $T > 0$ *such that*
$$0 < T < \min\left\{ \frac{-\min\{1, \frac{a}{2}\}}{\beta^2\left(\frac{a}{2} - BJ_{1/2} - CJ_{1/2}^2\right)}, \frac{-\min\{1, \frac{a}{2}\}}{\beta^2\left(\frac{a}{2} + bS - CJ_{-1/2}^2\right)} \right\}, \tag{2.4}$$
where β *is the immersion constant of* $H^1(0,T)$ *in* $C([0,T])$, $J_{1/2} := \frac{\sqrt{B^2+4AC}-b}{2c} + \frac{1}{2} > 0$, $J_{-1/2} := -\frac{\sqrt{B^2+4AC}+B}{2c} - 1/2 < 0$ *and* $S = \frac{\sqrt{b^2+2ac}+b}{2C} > 0$.

Then the equation
$$x'' - a(t)x + b(t)x^2 + c(t)x^3 = 0 \tag{2.5}$$
has at least two T- periodic solution.

3 Proof of Theorem 2.4

Proof. Consider the following Banach spaces

$$X = Y = \{x | x \in C(\mathbb{R}, \mathbb{R}), x(t+T) = x(t), \text{for all } t \in \mathbb{R}\},$$

with the norm

$$\|x\|_Y = |x|_\infty,$$

where $|x|_\infty = \max_{t \in [0,T]} |x(t)|$.

Define a linear operator $L : DomL \subset X \to Y$ by setting

$$DomL = \{x | x \in X, x'' \in C(\mathbb{R}, \mathbb{R})\}$$

and for $x \in DomL$,

$$Lx = x''. \tag{3.1}$$

We also define a nonlinear operator $N : Y \to Y$ by setting

$$Nx = a(t)x - b(t)x^2 - c(t)x^3. \tag{3.2}$$

It is not dificult to see that

$$KerL = \mathbb{R}, \text{ and } ImL = \left\{ x | x \in Y, \int_0^T x(s)ds = 0 \right\}.$$

Thus the operator L is a Fredholm operator with index zero.

Define the continuous projector $P : X \to KerL$ and the averaging projector $Q : Y \to Y$ by setting

$$Px(t) = x(0)$$

and

$$Qx(t) = \frac{1}{T} \int_0^T x(s)ds.$$

Hence, $ImP = KerL$ and $KerQ = ImL$. Denoting by $K_P : ImL \to DomL \cap KerP$ the inverse of $L|_{DomL \cap KerP}$, we have

$$K_P y(t) = \int_0^T G(s,t) y(s) ds,$$

where

$$G(s,t) = \begin{cases} -\dfrac{s}{T}[T-t] & , 0 \le s \le t \\ -\dfrac{t}{T}[T-s] & , t \le s \le T. \end{cases}$$

Then $QN : Y \to Y$ and $K_P(I-Q)N : X \to X$ read

$$QNx = \frac{1}{T}\int_0^T a(s)x(s)ds - \frac{1}{T}\int_0^T b(s)x^2(s)ds - \frac{1}{T}\int_0^T c(s)x^3(s)ds,$$

$$K_P(I-Q)Nx = \int_0^T G(s,t)a(s)x(s)ds - \int_0^T G(s,t)b(s)x^2(s)ds$$

$$-\int_0^T G(s,t)c(s)x^3(s)ds - QNx\left[\frac{t}{2}(t-T)\right].$$

Clearly, QN and $K_P(I-Q)N$ are continuous. By using Arzela-Ascoli theorem, it is not difficult to prove that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset Y$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset Y$.

Let

$$\Omega_{1/2} := \{x \in Y | M < x(t) < J_{1/2}\}, \tag{3.3}$$

an open set in Y, where

$$J_{1/2} := J + 1/2,$$
$$J := \frac{\sqrt{B^2+4AC}-b}{2c} > 0 \tag{3.4}$$

and

$$M := \frac{\sqrt{b^2+2ca}-B}{2C} > 0. \tag{3.5}$$

By (2.2) and (2.3), we have

$$M < \frac{\sqrt{b^2+4ac}-B}{2C} \leq \frac{\sqrt{b(t)^2+4a(t)c(t)}-b(t)}{2c(t)} \leq \frac{\sqrt{B^2+4AC}-b}{2c} < J_{1/2},$$

uniformly in t.

Note that

$$a(t) - b(t)J_{1/2} - c(t)J_{1/2}^2 < 0. \tag{3.6}$$

Indeed,

$$a(t) - b(t)J_{1/2} - c(t)J_{1/2}^2$$
$$= -c(t)\left(J_{1/2} + \frac{\sqrt{b(t)^2+4a(t)c(t)}+b(t)}{2c(t)}\right)\left(J_{1/2} - \frac{\sqrt{b(t)^2+4a(t)c(t)}-b(t)}{2c(t)}\right)$$
$$\leq -c(t)\left(J_{1/2} + \frac{\sqrt{b(t)^2+4a(t)c(t)}+b(t)}{2c(t)}\right)\frac{1}{2}$$
$$\leq -c\left(J_{1/2} + \frac{\sqrt{b^2+4ac}+b}{2C}\right)\frac{1}{2} < 0.$$

Note that

$$a(t) - b(t)M - c(t)M^2 > 0. \tag{3.7}$$

Indeed,

$$a(t) - b(t)M - c(t)M^2 > \frac{a(t)}{2} - b(t)M - c(t)M^2 =$$
$$= -c(t)\left(M + \frac{\sqrt{b(t)^2+2a(t)c(t)}+b(t)}{2c(t)}\right)\left(M - \frac{\sqrt{b(t)^2+2a(t)c(t)}-b(t)}{2c(t)}\right) \geq 0$$

Let $0 < \lambda < 1$ and x such that

$$x'' - \lambda a(t)x + \lambda b(t)x^2 + \lambda c(t)x^3 = 0.$$

Multiplying by x and integrand of 0 to T, we have that

$$\begin{aligned}0 &= \int_0^T (x')^2 + \lambda\frac{a(t)}{2}x^2 dt + \int_0^T \lambda\frac{a(t)}{2}x^2 - \lambda b(t)x^3 - \lambda c(t)x^4 dt \\ &= \int_0^T (x')^2 + \lambda\frac{a(t)}{2}x^2 dt + \int_0^T \lambda x^2(\frac{a(t)}{2} - b(t)x - c(t)x^2)dt\end{aligned}$$

By (2.4),(3.3) and (3.4), if $x \in \partial\Omega_{1/2}$, we have $M \leq |x|_\infty \leq J_{1/2}$ and by (2.2) and (3.6), $\frac{a}{2} - BJ_{1/2} - CJ_{1/2}^2 < 0$. Therefore

$$\begin{aligned}0 &\geq \lambda \min\{1, \frac{a}{2}\}\|x\|^2_{H^1(0,T)} + \lambda\int_0^T x^2\left(\frac{a}{2} - B|x|_\infty - C|x|_\infty^2\right)dt. \\ &\geq \lambda\left[\frac{\min\{1,\frac{a}{2}\}}{\beta^2}|x|_\infty^2 + \int_0^T x^2\left(\frac{a}{2} - BJ_{1/2} - CJ_{1/2}^2\right)dt\right] \\ &\geq \lambda\left[\frac{\min\{1,\frac{a}{2}\}}{\beta^2}|x|_\infty^2 + \left(\frac{a}{2} - BJ_{1/2} - CJ_{1/2}^2\right)T|x|_\infty^2\right] \\ &\geq \lambda\left[\frac{\min\{1,\frac{a}{2}\}}{\beta^2} + \left(\frac{a}{2} - BJ_{1/2} - CJ_{1/2}^2\right)T\right]|x|_\infty^2 > 0,\end{aligned}$$

where β is the immersion constant of $H^1(0,T)$ in $C([0,T])$. But this is a contradiction. Therefore the condition (1) of Proposition 2.3 holds for $\Omega_{1/2}$.

Take $x \in \partial\Omega_{1/2} \cap KerL$, we have, $x = M$ or $x = J_{1/2}$.

If $x = J_{1/2}$, by (3.6), we have

$$a(t) - b(t)x - c(t)x^2 < 0.$$

If $x = M$, by (3.7), we have

$$a(t) - b(t)x - c(t)x^2 > 0.$$

Then, for each $x \in \partial\Omega_{1/2} \cap KerL$, we have that

$$QNx = \frac{1}{T}\int_0^T x\left(a(t) - b(t)x - c(t)x^2\right)dt \neq 0. \tag{3.8}$$

Therefore the condition (2) of Proposition 2.3 holds for $\Omega_{1/2}$.

Let $\frac{M+J_{1/2}}{2}$, the arithmetic mean between M and $J_{1/2}$. Furthermore, define a continuous function $H(x,\mu)$ by setting

$$H(x,\mu) = -(1-\mu)(x - \frac{M+J_{1/2}}{2}) + \mu\frac{1}{T}\int_0^T x\left(a(t) - b(t)x - c(t)x^2\right)dt, \mu \in [0,1].$$

It follows from (3.14) that

$$H(x,\mu) \neq 0, \text{ for all } x \in \partial\Omega_{1/2} \cap KerL.$$

Hence, using the homotopy invariance theorem, we have

$$deg(QN, \Omega_{1/2} \cap KerL, 0) = deg\left(\frac{1}{T}\int_0^T x\left(a(t) - b(t)x - c(t)x^2\right)dt, \Omega_{1/2} \cap KerL, 0\right)$$

$$= deg\left(-(x - \frac{M + J_{1/2}}{2}), \Omega_{1/2} \cap KerL, 0\right) = -1 \neq 0.$$

In view of all the discussions above, we conclude from Proposition 2.3 that the equation (2.5) has a solution in $\bar{\Omega}_{1/2}$.

Now, we will prove the existence of a second solution for equation (2.5). By (2.2), we have

$$\frac{\sqrt{b^2 + 2ac} + b}{2C} \leq \frac{\sqrt{b(t)^2 + 2a(t)c(t)} + b(t)}{2c(t)}$$
$$< \frac{\sqrt{b(t)^2 + 4a(t)c(t)} + b(t)}{2c(t)} \leq \frac{\sqrt{B^2 + 4AC} + B}{2c}$$
$$< \frac{\sqrt{B^2 + 4AC} + B}{2c} + 1/2.$$

Ready

$$-\frac{\sqrt{B^2 + 4AC} + B}{2c} - 1/2 < \frac{-\sqrt{b(t)^2 + 4a(t)c(t)} - b(t)}{2c(t)} < -\frac{\sqrt{b^2 + 2ac} + b}{2C},$$

uniformly in t.

Let

$$\Omega_{-1/2} := \{x \in Y | J_{-1/2} < x(t) < -S\}, \tag{3.9}$$

an open set in Y, where

$$J_{-1/2} := -\frac{\sqrt{B^2 + 4AC} + B}{2c} - 1/2$$

and

$$S = \frac{\sqrt{b^2 + 2ac} + b}{2C} \tag{3.10}$$

Note that

$$a(t) - b(t)J_{-1/2} - c(t)J_{-1/2}^2 < 0. \tag{3.11}$$

and

$$a(t) + b(t)S - c(t)S^2 > 0. \tag{3.12}$$

Let $0 < \lambda < 1$ and x such that

$$x'' - \lambda a(t)x + \lambda b(t)x^2 + \lambda c(t)x^3 = 0.$$

Multiplying by x and integrand of 0 to T, we have that

$$0 = \int_0^T (x')^2 + \lambda \frac{a(t)}{2} x^2 dt + \int_0^T \lambda \frac{a(t)}{2} x^2 - \lambda b(t)x^3 - \lambda c(t)x^4 dt$$
$$= \int_0^T (x')^2 + \lambda \frac{a(t)}{2} x^2 dt + \int_0^T \lambda x^2 (\frac{a(t)}{2} - b(t)x - c(t)x^2) dt$$

By (2.2) and (3.9), if $x \in \partial\Omega_{-1/2}$, we have $J_{-1/2} \leq x \leq -S$ and

$$0 \geq \lambda \min\{1, \frac{a}{2}\} \|x\|_{H^1(0,T)}^2 + \lambda \int_0^T x^2 (\frac{a}{2} + bS - CJ_{-1/2}^2) dt. \tag{3.13}$$

By (3.9) and (3.11), we have that

$$\frac{a}{2} + bS - CJ_{-1/2}^2 \leq \frac{a}{2} - bJ_{-1/2} - CJ_{-1/2}^2 \leq a(t) - b(t)J_{-1/2} - c(t)J_{-1/2}^2 < 0.$$

Therefore, it follows from (3.13) and (2.4)

$$\begin{aligned} 0 &\geq \lambda \left[\frac{\min\{1, \frac{a}{2}\}}{\beta^2} |x|_\infty^2 + \int_0^T x^2 \left(\frac{a}{2} + bS - CJ_{-1/2}^2 \right) dt \right] \\ &\geq \lambda \left[\frac{\min\{1, \frac{a}{2}\}}{\beta^2} |x|_\infty^2 + \left(\frac{a}{2} + bS - CJ_{-1/2}^2 \right) T |x|_\infty^2 \right] \\ &\geq \lambda \left[\frac{\min\{1, \frac{a}{2}\}}{\beta^2} + \left(\frac{a}{2} + bS - CJ_{-1/2}^2 \right) T \right] |x|_\infty^2 > 0, \end{aligned}$$

where β is the immersion constant of $H^1(0,T)$ in $C([0,T])$. But this is a contradiction. Therefore the condition (1) of Proposition 2.3 holds for $\Omega_{-1/2}$.

Take $x \in \partial\Omega_{-1/2} \cap KerL$, we have, $x = J_{-1/2}$ or $x = -S$. By (3.11) and (3.12) we have

$$QNx = \frac{1}{T} \int_0^T x \left(a(t) - b(t)x - c(t)x^2 \right) dt \neq 0. \tag{3.14}$$

Therefore the condition (2) of Proposition 2.3 holds for $\Omega_{-1/2}$.

Let $\frac{J_{-1/2}-S}{2}$, the arithmetic mean between $J_{-1/2}$ and $-S$. Furthermore, define a continuous function $H(x,\mu)$ by setting

$$H(x,\mu) = -(1-\mu)(x - \frac{J_{-1/2} - S}{2}) + \mu \frac{1}{T} \int_0^T x \left(a(t) - b(t)x - c(t)x^2 \right) dt, \mu \in [0,1].$$

It follows from (3.14) that

$$H(x,\mu) \neq 0, \text{ for all } x \in \partial\Omega_{-1/2} \cap KerL.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned} &deg(QN, \Omega_{-1/2} \cap KerL, 0) = \\ &\quad deg\left(\frac{1}{T} \int_0^T x \left(a(t) - b(t)x - c(t)x^2 \right) dt, \Omega_{-1/2} \cap KerL, 0 \right) \\ &\quad = deg\left(-\left(x - \frac{J_{-1/2} - S}{2} \right), \Omega_{-1/2} \cap KerL, 0 \right) = -1 \neq 0. \end{aligned}$$

In view of all the discussions above, we conclude from Proposition 2.3 that the equation (2.5) has a solution in $\bar{\Omega}_{-1/2}$.

Since $\bar{\Omega}_{1/2} \cap \bar{\Omega}_{-1/2} = \emptyset$, the Theorem 2.4 is proved. ∎

4 Positive Homoclinic Solutions

In this section we consider the existence of positive solutions of the problem

$$x'' - a(t)x + b(t)x^2 + c(t)x^3 = 0, t \in \mathbb{R} \tag{4.1}$$

$$x(\pm\infty) = x'(\pm\infty) = 0.$$

Suppose $a(t), b(t)$ and $c(t)$ are continuous 2π-periodic functions, subject to the constraints

$$0 < a \leq a(t), 0 \leq b(t) \leq B, 0 < c \leq c(t) \leq C, \tag{4.2}$$

for constants $a, B, c, C > 0$.

For each $n \in \mathbb{N}$, we consider the periodic problem

$$x'' - a(t)x + b(t)x^2 + c(t)x^3 = 0, t \in (-n\pi, n\pi) \tag{4.3}$$

$$x(-n\pi) = x(n\pi).$$

Set $I_n = [-n\pi, n\pi]$ and consider the Sobolev space

$$H_n = \{x \in H^1(I_n) : x(-n\pi) = x(n\pi)\},$$

with the norm

$$\|x\|_n = \left(\int_{-n\pi}^{n\pi} [x^2(t) + x'^2(t)]dt \right)^{1/2}.$$

We solve (4.3) using mountain pass theorem and obtain uniform estimates for their solutions. We need the technical result contained in P. Rabinowitz [9].

Proposition 4.1. *Let $x \in H^1_{loc}(\mathbb{R})$. Then*

(i) *If $T \geq 1$, for $t \in [T-1, T+1]$,*

$$\max_{t \in [T-1, T+1]} |x(t)| \leq 2 \left(\int_{T-1}^{T+1} [x^2(s) + x'^2(s)]ds \right)^{1/2}. \tag{4.4}$$

(ii) *For every $x \in H_n$,*

$$\|x\|_{L^\infty(-n\pi, n\pi)} \leq 2\|x\|_n. \tag{4.5}$$

Proposition 4.2. *Let assumptions (4.2) hold. Then, for every $n \in \mathbb{N}$, the problem (4.3) has a positive solution x_n. Moreover, there is a constant $K > 0$, independently of n, such that*

$$\|x_n\|_n \leq K. \tag{4.6}$$

Proof. Consider the modified problem

$$x'' - a(t)x + b(t)x^2 + c(t)x_+^3 = 0, t \in (-n\pi, n\pi) \tag{4.7}$$

$$x(-n\pi) = x(n\pi),$$

where $x_+ = \max(x, 0)$. It is easy to see that solutions of (4.7) are positive solutions of (4.3). Indeed, if $x(t)$ is a solution of (4.7) and $x(t)$ has a negative minimum at $t_0 \in (-n\pi, n\pi)$ by the equation, we derive the contradiction

$$0 = x''(t_0) - a(t_0)x(t_0) + b(t_0)x(t_0)^2 > 0.$$

Then $x(t) \geq 0$ and x is a solution of (4.3).

To prove the existence of a solution of (4.7). we consider the functional $f_n : H_n \to \mathbb{R}$ defined by

$$f_n(x) = \int_{-n\pi}^{n\pi} \left(\frac{1}{2}(x'^2 + a(t)x^2) - \frac{1}{3}b(t)x^3 - \frac{1}{4}c(t)x_+^4 \right) dt.$$

Critical points of f_n are weak solutions of (4.7) and, in a standard way, classical solutions. The proof that f_n satisfies the assumptions of the mountain pass theorem and the inequality (4.6) follows from Grossinho, Minhóz and Tersian [5, Proposition 2].

Obviously $f_n(0) = 0$. By assumption (4.2) it follows that

$$f_n(x) \geq \frac{\widehat{a}}{4}\|x\|_n^2 + \int_{-n\pi}^{n\pi} x^2 \left(\frac{1}{4}a - \frac{1}{3}B|x| - \frac{1}{4}C|x|^2 \right) dt.$$

The continuous function $\phi(t) = \frac{1}{4}a - \frac{1}{3}Bt - \frac{1}{4}Ct^2$ is such that $\phi(0) = \frac{1}{4}a > 0$ and $\lim_{t \to +\infty} \phi(t) = -\infty$. Then, there is $\rho > 0$ such that $\phi(\rho) = 0$.

By (4.5), if $\|x\|_n = \rho > 0$ then $f_n(x) \geq \frac{\widehat{a}}{4}\rho^2 > 0$.

Let $x_0 \in H_1$ be such that $x_0(t) > 0$ if $t \in (-1, 1)$ and $x_0(-1) = x_0(1) = 0$. Consider the function

$$\hat{x_0}(t) = \begin{cases} \lambda x_0(t), & \text{if } \ t \in [-1, 1], \\ 0 & \text{if } \ t \in [-n\pi, n\pi] - [-1, 1]. \end{cases}$$

It is easy to see that $f_n(\hat{x_0}) < 0$ for λ large enough. By the mountain pass theorem, there exists a solution $x_n \in H_n$ such that

$$c_n = f_n(x_n) = \inf_{\gamma \in \Gamma_n} \max_{t \in [0,1]} f_n(\omega(t)), \;\; f_n'(x_n) = 0, \tag{4.8}$$

where

$$\Gamma_n = \{\omega(t) \in C([0, 1], H_n) : \omega(0) = 0, \omega(1) = \hat{x_0}(t)\}.$$

Moreover, using the variational characterization (4.8), we have

$$c_n \geq \frac{\widehat{a}}{4}\rho^2 > 0.$$

Therefore, x_n is a classical, nontrivial, and nonnegative solution of (4.3). ∎

Theorem 4.3. *Let assumptions (4.2) hold. Then the problem (4.1) has a positive homoclinic solutions.*

Proof. For every $n \in \mathbb{N}$, consider the solution x_n of problem (4.3), given by Proposition 4.2. By (4.6), and the embedding of H_n in $C[-n\pi, n\pi]$, there is K_1 such that $\|x_n\|_{C^0} \leq K_1$. Then, by equation (4.3), it follows that $\|x_n''\|_{C^0} \leq K_2$, which easily implies $\|x_n\|_{C^2} \leq K$, where K_1, K_2 , and K are positive constants independent of n. Consider the periodic extension of u_n to $\mathbb{R}$ and denote it by the same symbol. Then u_n is a $2n\pi$-periodic solution of (4.1). By the bounds obtained above and (4.1) we can derive that there exists a subsequence of (x_n) which converges in $C^2_{loc}(\mathbb{R})$ to a solution x of (4.1) that satisfies

$$\int_{-\infty}^{+\infty} (x'^2 + x^2) dt < \infty. \tag{4.9}$$

It remains to show that x is nonzero and $x(\pm\infty) = x'(\pm\infty) = 0$. Let $t_n \in [-n\pi, n\pi]$ be a point where u_n attains its maximum value. Since $x_n(t_n) > 0$ and $x_n''(t_n) \leq 0$, it follows by (4.1) that

$$x_n(t_n)(-a(t_n) + b(t_n)x_n(t_n) + c(t_n)x_n^2(t_n)) = -x_n''(t_n) \geq 0.$$

Then, by assumptions (4.2),

$$\begin{aligned} x_n(t_n) &\geq \frac{-b(t_n) + \sqrt{b^2(t_n) + 4a(t_n)c(t_n)}}{2c(t_n)} \\ &\geq \frac{-b(t_n) + \sqrt{b^2(t_n) + 4ac(t_n)}}{2c(t_n)} \\ &= \frac{4a(t_n)c(t_n)}{2c(t_n)[b(t_n) + \sqrt{b^2(t_n) + 4ac(t_n)}]} \\ &\geq \frac{4ac}{2C[B + \sqrt{B^2 + 4aC}]} = C_3 > 0, \end{aligned} \tag{4.10}$$

independently of n.

The functionals f_n are invariant by translations of t by integer multiples of 2π, therefore as $x_n(t_n) \geq c > 0$ for almost all n, we can assume that there exists $t_n \to t_0$ in $[\pi, \pi]$. By the uniform convergence of (x_n) in $[\pi, \pi]$ and by (4.10), it follows that $x(t_0) \geq c > 0$ and x is a nontrivial and nonnegative solution of (4.1).

By (4.9) and Proposition 4.5 it follows that

$$\lim_{T\to\pm\infty} \max_{t\in[T-1,T+1]} |x(t)| \leq \lim_{T\to\pm\infty} \int_{T-1}^{T+1} (x'(t)^2 + x^2(t)) dt = 0, \tag{4.11}$$

so $x(\pm\infty) = 0$.

Next we prove that $x'(\pm\infty) = 0$. By assumption (4.2) there exists $M > 0$ such that $|x''(t)| \leq M$ in $\mathbb{R}$. Suppose, by contradiction, $x'(+\infty) \neq 0$. Then, there exists

$\epsilon > 0$ and a sequence $\tau_n \to +\infty$ such that $|x'(\tau_n)| \geq \epsilon$ for all n. By the mean value theorem, for $t \in (\tau_n - \delta, \tau_n + \delta)$, where $\delta \in (0, \frac{\epsilon}{2M})$,

$$|x'(t)| \geq |x'(\tau_n)| - |x'(\tau_n) - x'(t)| \geq \epsilon - |x''(\xi_n)||\tau_n - t| \geq \epsilon - M\delta \geq \frac{\epsilon}{2}.$$

Therefore

$$\int_{\tau_n-\delta}^{\tau_n+\delta} x'^2(t)dt \geq \frac{\delta\epsilon^2}{2},$$

which is a contradiction to the equality of (4.11).

The case $x'(-\infty) = 0$ is analogous. ■

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