Gregus type fixed point results for tangential mappings satisfying contractive conditions of integral type*

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Abstract

The notion of pair-wise tangential mappings, which is a generalization of mappings satisfying (E.A) property, is introduced and used to prove a common fixed point theorem of Gregus type for a quadruple of self mappings of a metric space satisfying a strict general contractive condition of integral type. Our main result generalizes a recent result of A. Djoudi, A. Alioche [Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type, J. Math. Anal. Appl. 329 (2007), 31-45].

1 Introduction and Preliminaries

Let A and B be two self-maps of a metric space $X = (X, d)$. The pair (A, B) is called (1) *commuting* if $ABx = BAx$ for all $x \in X$; (2) *weakly commuting* (Sessa [12]) if $d(ABx, BAx) \leq d(Ax, Bx)$ for all $x \in X$; (3) *compatible* (Jungck [5]) if $lim_n\, d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim_n Ax_n = lim_n Bx_n = z$, for some $z \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting maps are compatible but neither implication is reversible (see, e.g., Example 1 of Sessa and Fisher [13] and Example 2.2 of Jungck

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[5]). The pair (A, B) is said to be (4) *weakly compatible* (Jungck [6]) if $ABx = BAx$ whenever $Ax = Bx$; (5) R-weakly commuting (Pant [8]) at a point $x \in X$ if for some $R > 0$ such that $d(ABx, BAx) \leq R\, d(Ax, Bx)$. It was proved in [9] that pointwise R-weak commutativity is equivalent to commutativity at a coincedence points; i.e., (A, B) is pointwise R-weak commuting if and only if (A, B) is weakly compatible.

M. Aamri and D. El Moutawakil [1] defined property (E.A) as follows.

Definition 1.1. The pair (A, B) satisfies property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Bx_n = z \in X. \tag{1}$$

In 2000, Sastry and Krishna Murthy [11] introduced the following notions:
A point $z \in X$ is said to be a *tangent point* to (A, B) if there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Bx_n = z$. (A, B) is called *tengential* if there exists $z \in X$, which is tangent to (A, B). Two year later, Aamri and El-Moutawakil rediscovered this notion and called it as property (E.A) (It seems that they were unaware of [11])

It is clear from the definition of compatibility that the pair (A, B) is noncompatible (see also, Pant [10]) if there exists at least one sequence $\{x_n\}$ in X such that (1) holds but, $lim_n\, d(ABx_n, BAx_n)$ is either nonzero or does not exist.

Recently, Liu et al. [7] defined a common property (E.A) as follows.

Definition 1.2. Let $A, B, S, T : X \to X$ be mappings. Then the pairs (A, S) and (B, T) satisfy a common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z \in X. \tag{2}$$

If $B = A$ and $T = S$ in (2), we obtain the definition of property (E.A).

In 2003, Djoudi and Nisse [4] proved the following theorem.

Theorem A. Let A, B, S and T be mappings from a Banach space X into itself satisfying

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{3}$$

$$\begin{aligned} \|Ax - By\|^p \leq \varphi(a\,\|Sx - Ty\|^p + (1-a)\, max\{&\alpha\,\|Ax - Sx\|^p, \beta\,\|By - Ty\|^p, \\ &\|Ax - Sx\|^{\frac{p}{2}} \cdot \|Ax - Ty\|^{\frac{p}{2}}, \|Ax - Ty\|^{\frac{p}{2}} \cdot \|Sx - By\|^{\frac{p}{2}}, \\ &\frac{1}{2}(\|Ax - Sx\|^p + \|By - Ty\|^p)\}) \end{aligned} \tag{4}$$

for all x, y in X, where $0 < a \leq 1, 0 < \alpha, \beta \leq 1, p \geq 1$ and $\varphi \in \mathcal{F}$. If $A(X)$ or $B(X)$ is closed and the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Inspiring from the recent results of Branciari [2] and Vijayaraju at el. [14], Djoudi and Alioche [3] proved the following theorems for mappings satisfying a general contractive condition of integral type.

Theorem B. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying (3) and

$$\Big(\int_0^{d(Ax,By)} \psi(t)\,dt\Big)^p \leq \varphi\Big[a\,\Big(\int_0^{d(Sx,Ty)} \psi(t)\,dt\Big)^p + (1-a)\,max\Big\{\int_0^{d(Ax,Sx)} \psi(t)\,dt, \int_0^{d(By,Ty)} \psi(t)\,dt,\ \Big(\int_0^{d(Ax,Sx)} \psi(t)\,dt\Big)^{\frac{1}{2}} \cdot \Big(\int_0^{d(Ax,Ty)} \psi(t)\,dt\Big)^{\frac{1}{2}}, \Big(\int_0^{d(Sx,By)} \psi(t)\,dt\Big)^{\frac{1}{2}} \cdot \Big(\int_0^{d(Ax,Ty)} \psi(t)\,dt\Big)^{\frac{1}{2}}\Big\}^p\Big] \tag{5}$$

for all x, y in X, where $0 < a \leq 1, p \geq 1$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue integrable mapping which is summable nonnegative and such that

$$\int_0^{\epsilon} \psi(t)\,dt > 0 \ \text{ for } \ \text{each } \epsilon > 0. \tag{6}$$

Suppose that one of $S(X)$ or $T(X)$ is complete and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

Theorem C. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying (3) and

$$\int_0^{d(Ax,By)} \psi(t)\,dt < a\int_0^{d(Sx,Ty)} \psi(t)\,dt + (1-a)\,max\Big\{\int_0^{d(Ax,Sx)} \psi(t)\,dt, \int_0^{d(By,Ty)} \psi(t)\,dt,\ \Big(\int_0^{d(Ax,Sx)} \psi(t)\,dt\Big)^{\frac{1}{2}} \cdot \Big(\int_0^{d(Ax,Ty)} \psi(t)\,dt\Big)^{\frac{1}{2}}, \Big(\int_0^{d(Sx,By)} \psi(t)\,dt\Big)^{\frac{1}{2}} \cdot \Big(\int_0^{d(Ax,Ty)} \psi(t)\,dt\Big)^{\frac{1}{2}}\Big\} \tag{7}$$

for all x, y in X for which the right-hand side of (7) is positive, where $0 < a < 1$ and ψ satisfies (6). Suppose that (A, S) or (B, T) satisfies property (E.A), one of $A(X), B(X), S(X), T(X)$ is a closed subspace of X and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

Now there arises a natural question-Is it possible to remove/or weaken the following conditions in Theorem C:
(i) the inclusion conditions (3);
(ii) the property (E.A) of the pairs (A, S) or (B, T);
(iii) the property of closedness of one of $A(X), B(X), S(X), T(X)$?

We give an affirmative answer to this question in our main result (see, Theorem 2.5 below).

Our main objective of this paper is to define the notion of pair-wise tangential mappings and to prove a common fixed point theorem of Gregus type for a quadruple of such self mappings of a metric space satisfying a strict general contractive condition of integral type.

2 Main results

We first introduce the concepts of *weak tangent point* for a pair of mappings and *pair-wise tangential* property for a dual pair of mappings.

Definition 2.1. Let $A, B, S, T : X \to X$ be mappings. A point $z \in X$ is said to be a *weak tangent point* to (S, T) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Ty_n = z$. The pair (A, B) is called *tangential* w.r.t the pair (S, T) if

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} By_n = z \tag{8}$$

whenever there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z$ (i.e., z is a weak tangent point to (S, T).

Special cases: (i) If $B = A$ and $T = S$ in (8), we say that the mapping A is *tangential* w.r.t the mapping S.
(ii) If $S = A$ and $T = B$ in (8), we say that (A, B) is *tangential* with itself.

Clearly, every pair of mappings (S, T) satisfies property (E.A) also has a point z in X which is tangent to (S, T) (to see this, just take $\{y_n\} = \{x_n\}$), but the converse need not be true (see, for instance, Example 2.2 below).

Let $\mathbb{R}_+$ be the set of nonnegative real numbers and $\mathbb{N}$ the set of natural numbers. Throughout this section, let $\mathcal{F}$ be the family of mappings φ from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that each φ is upper semicontinuous, nondecreasing and $\varphi(t) < t$ for all $t > 0$.

Example 2.2. Let $X = (\mathbb{R}_+, d)$ be the metric space endowed with usual metric d. Let $A, B, S, T : \mathbb{R}_+ \to \mathbb{R}_+$ be mappings defined by

$$Ax = x + 1, Bx = x + 2, Sx = x + 3 \text{ and } Tx = x + 4 \text{ for all } x \text{ in } X.$$

Take two sequences $\{x_n = 2+\frac{1}{n}\}$ and $\{y_n = 1+\frac{1}{n}\}$ in $\mathbb{R}_+$. Clearly, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = 3$; i.e., $3 \in \mathbb{R}_+$ is a weak tangent point to (A, B). But, there exists no sequence $\{x_n\}$ in $\mathbb{R}_+$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z in $\mathbb{R}_+$. It follows that the pair (A, B) fails to satisfy property (E.A). Let us consider another pair of sequences $\{x_n = 1 - \frac{1}{n}\}$ and $\{y_n = \frac{1}{n}\}$ in $\mathbb{R}_+$. Then we see that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = 4$; i.e., $4 \in \mathbb{R}_+$ is a weak tangent point to (S, T). But, there exists no sequence $\{x_n\}$ in $\mathbb{R}_+$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in $\mathbb{R}_+$. It follows that the pair (S, T) fails to satisfy property (E.A). Note also that the mappings A, B, S, T do not satisfy common (E.A) property.

Hence, we conclude that every pair of mappings (S, T) which satisfies property (E.A)(or has a tangent point) also has a *weak tangent point* to (S, T), but the converse is not necessarily true. Hence, our notion of *weak tangent point* to the pair (S, T) is weaker than the notion of property (E.A) of the pair (S, T) (and the notion of tangent point to (S, T)).

It may be remarked that if the pair (A, B) is *tangential* w.r.t the pair (S, T), then the pair (S, T) need not be *tangential* w.r.t the pair (A, B). However, if the pair (A, B) is *tangential* w.r.t the pair (S, T), and if the pair (S, T) is *tangential* w.r.t the pair (A, B), then the pairs (A, S) and (B, T) satisfy a common property (E.A).

Now we show in Example 2.3 below that the pair (A, B) is *tangential* w.r.t the pair (S, T), but the pair (S, T) is not *tangential* w.r.t the pair (A, B).

Example 2.3. Let $X = (\mathbb{R}_+, d)$ be the metric space endowed with usual metric d. Let $A, B, S, T : \mathbb{R}_+ \to \mathbb{R}_+$ be mappings defined by

$Ax = 1$, if $x < 1$ and $Ax = \frac{x^2}{3}$, if $x \geq 1$;
$Bx = x^3$ if $x \leq \frac{1}{2}$, $Bx = \frac{2}{3}$ if $\frac{1}{2} < x < 1$ and $Bx = 1$ if $x \geq 1$;
$Sx = 1$, if $x \leq 1$ and $Sx = \frac{x^3}{2}$, if $x > 1$, and
$Tx = 0$ if $x \leq 1$, $Tx = 1$ if $1 < x \leq 2$ and $Tx = \frac{1}{3}$ if $x > 2$.

Clearly, there exist two sequences $\{x_n = \frac{1}{n+1}\}$ and $\{y_n = 1 + \frac{1}{n+1}\}$ in $\mathbb{R}_+$ such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Ty_n = 1 \in \mathbb{R}_+$. Then $lim_{n\to\infty} Ax_n = lim_{n\to\infty} By_n = 1$. Also, $lim_{n\to\infty} Sx_n = 1$, $lim_{n\to\infty} Tx_n = 0$. On the other hand, there exist two sequences $\{x_n = \frac{1}{n+1}\}$ and $\{y_n = (n+1)^2\}$ in $\mathbb{R}_+$ such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} By_n = 1 \in \mathbb{R}_+$, but $lim_{n\to\infty} Sx_n = 1$, and $lim_{n\to\infty} Ty_n = \frac{1}{3} \neq 1$. Also, $lim_{n\to\infty} Ax_n = 1, lim_{n\to\infty} Bx_n = 0 \in \mathbb{R}_+$.

Remark 2.4. We observe the following facts from Example 2.3 above:
(i) 1 in $\mathbb{R}_+$ is a *weak tangent point* to both the pairs (A, B) and (S, T);
(ii) the pair (A, B) is tangential w.r.t the pair (S, T), but the pair (S, T) is not tangential w.r.t the pair (A, B);
(iii) the mappings A, B, S, T satisfy common (E.A) property.

Note also in Example 2.3 above that $A(X) = [\frac{1}{3}, \infty)$, $S(X) = [1, \infty)$ and $B(X) = [0, \frac{1}{8}] \cup \{\frac{2}{3}, 1\}$, $T(X) = \{0, \frac{1}{3}, 1\}$. Thus, $A(X) \nsubseteq T(X)$ and $B(X) \nsubseteq S(X)$. However, $S(X) \cap T(X) = \{1\}$, $A(X) \cap T(X) = \{\frac{1}{3}, 1\}$ and $B(X) \cap S(X) = \{1\}$.

Example 2.5. Let $(\mathbb{R}_+, d)$ be the metric space endowed with usual metric d. Let $A, B, S, T : \mathbb{R}_+ \to \mathbb{R}_+$ be mappings defined by

$Ax = |sin\, x|$, if $x < 1$ and $Ax = 1$, if $x \geq 1$;
$Bx = 1 - |cos\, x^4|$ if $x < 1$ and $Bx = cos\, 1$ if $x \geq 1$;
$Sx = 1$, if $x < 1$, $Sx = |sin\, 2\pi x|$, if $x \geq 1$; and
$Tx = 1 - |cos\, 2\pi x|$ for all $x \in \mathbb{R}_+$.

Clearly, there exist two sequences $\{x_n = n\}$ and $\{y_n = n^2\}$ in $\mathbb{R}_+$ such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Ty_n = 0 \in \mathbb{R}_+$, but $lim_{n\to\infty} Ax_n = 1$ and $lim_{n\to\infty} By_n = cos\, 1$. Also, $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = 0$. Thus, 0 in $\mathbb{R}_+$ is a weak tangent point to the pair (S, T) and the pair (S, T) also satisfies property (E.A), but the pair (A, B) is not tangential to the pair (S, T). On the other hand, there exist two sequences $\{x_n = \frac{1}{n+1}\}$ and $\{y_n = \frac{1}{(n+1)^2}\}$ in $\mathbb{R}_+$ such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} By_n = 0 \in \mathbb{R}_+$, but $lim_{n\to\infty} Sx_n = 1$, and $lim_{n\to\infty} Ty_n = 0$. Also, $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Bx_n = 0$. Thus, 0 in $\mathbb{R}_+$ is a weak tangent point to the pair (A, B) and the pair (A, B) also satisfies property (E.A), but the pair (S, T) is not tangential w.r.t the pair (A, B). Note also that there exist no two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}_+$ such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} By_n = lim_{n\to\infty} Sx_n = lim_{n\to\infty} Ty_n = t \in \mathbb{R}_+$. Hence, the mappings A, B, S, T do not satisfy common (E.A) property.

A sketch of downward trend of implications (from stronger to weaker conditions) follow from the respective definitions of noncompatibility, (E.A) property, a common (E.A) property and tangential property are shown in Fig.1 below:

[the pair (A, B) is tangential w.r.t the pair (S, T)]
$\not\Downarrow$ *or* $\not\Uparrow$
$\not\Uparrow$ $\Downarrow$ [the pair (S, T) is tangential w.r.t the pair (A, B)]
$\Downarrow$ $\not\Uparrow$
[pairs (A, S) and (B, T) have a common (E.A) property]
$\Downarrow$ $\not\Uparrow$
[noncompatibility of (A, S)] $\Rightarrow$ [(A, S) has (E.A) property] $\Downarrow$ $\not\Uparrow$
and
$\Downarrow$ [(B, T) has (E.A) property]
[a point z in X is a weak tangent point to (A, S) $\Downarrow$
[a point z in X is a weak tangent point to (B, T)]

Fig.1

Now we state and prove our main result.

Theorem 2.5. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying

$$[1+\alpha\int_0^{d(Sx,Ty)}\psi(t)\,dt]\int_0^{d(Ax,By)}\psi(t)\,dt < \alpha[\int_0^{d(Ax,Sx)}\psi(t)\,dt$$
$$\cdot\int_0^{d(By,Ty)}\psi(t)\,dt + \int_0^{d(Ax,Ty)}\psi(t)\,dt\cdot\int_0^{d(Sx,By)}\psi(t)\,dt]$$
$$+a\int_0^{d(Sx,Ty)}\psi(t)\,dt + (1-a)\,max\Big\{\int_0^{d(Ax,Sx)}\psi(t)\,dt,$$
$$\int_0^{d(By,Ty)}\psi(t)\,dt,\ (\int_0^{d(Ax,Sx)}\psi(t)\,dt)^{\frac{1}{2}}\cdot(\int_0^{d(Ax,Ty)}\psi(t)\,dt)^{\frac{1}{2}},$$
$$(\int_0^{d(Sx,By)}\psi(t)\,dt)^{\frac{1}{2}}\cdot(\int_0^{d(Ax,Ty)}\psi(t)\,dt)^{\frac{1}{2}}\Big\} \tag{9}$$

for all x, y in X for which the right-hand side of (9) is positive, where $0 < a < 1$, $\alpha \geq 0$ and ψ satisfies (6). Suppose that one of the following conditions (a)-(c) holds:
(a) there is a weak tangent point $z \in S(X) \cap T(X)$ to (S, T) and (A, B) is tangential w.r.t (S, T),
(b) there is a weak tangent point $z \in A(X) \cap T(X)$ to (A, T) and (S, B) is tangential w.r.t (A, T),
(c) there is a weak tangent point $z \in B(X) \cap S(X)$ to (S, B) and (A, T) is tangential w.r.t (S, B);
and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

Proof. Suppose (a) holds. Since a point $z \in S(X) \cap T(X)$ is a weak tangent point to (S, T), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty}Sx_n = lim_{n\to\infty}Ty_n = z$. Because the pair of mapping (A, B) is tangential w.r.t the pair (S, T), we have

$$lim_{n\to\infty}Ax_n = lim_{n\to\infty}By_n = z.$$

Again, since $z \in S(X) \cap T(X)$, $z = Su = Tv$ for some $u, v \in X$. If $Bv \neq z$, using (9) we get

$$[1+\alpha\int_0^{d(Sx_n,Tv)}\psi(t)\,dt]\int_0^{d(Ax_n,Bv)}\psi(t)\,dt < \alpha[\int_0^{d(Ax_n,Sx_n)}\psi(t)\,dt$$
$$\cdot\int_0^{d(Bv,Tv)}\psi(t)\,dt + \int_0^{d(Ax_n,Tv)}\psi(t)\,dt\cdot\int_0^{d(Sx_n,Bv)}\psi(t)\,dt]$$
$$+a\int_0^{d(Sx_n,Tv)}\psi(t)\,dt + (1-a)\,max\Big\{\int_0^{d(Ax_n,Sx_n)}\psi(t)\,dt,$$
$$\int_0^{d(Bv,Tv)}\psi(t)\,dt,\ (\int_0^{d(Ax_n,Sx_n)}\psi(t)\,dt)^{\frac{1}{2}}\cdot(\int_0^{d(Ax_n,Tv)}\psi(t)\,dt)^{\frac{1}{2}},$$

$$(\int_0^{d(Sx_n,Bv)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax_n,Tv)} \psi(t)\,dt)^{\frac{1}{2}}\Big\}.$$

Letting $n \to \infty$ we obtain

$$\int_0^{d(z,Bv)} \psi(t)\,dt \leq (1-a) \int_0^{d(z,Bv)} \psi(t)\,dt < \int_0^{d(z,Bv)} \psi(t)\,dt,$$

which is a contradiction. Thus, $Bv = z$.

Further, if $Au \neq z$, using (9) again, we get

$$[1 + \alpha \int_0^{d(Su,Ty_n)} \psi(t)\,dt] \int_0^{d(Au,By_n)} \psi(t)\,dt < \alpha[\int_0^{d(Au,Su)} \psi(t)\,dt$$
$$\cdot \int_0^{d(By_n,Ty_n)} \psi(t)\,dt + \int_0^{d(Au,Ty_n)} \psi(t)\,dt \cdot \int_0^{d(Su,By_n)} \psi(t)\,dt]$$
$$+a \int_0^{d(Su,Ty_n)} \psi(t)\,dt + (1-a)\, max\Big\{ \int_0^{d(Au,Su)} \psi(t)\,dt,$$
$$\int_0^{d(By_n,Ty_n)} \psi(t)\,dt,\ (\int_0^{d(Au,Su)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Au,Ty_n)} \psi(t)\,dt)^{\frac{1}{2}},$$
$$(\int_0^{d(Su,By_n)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Au,Ty_n)} \psi(t)\,dt)^{\frac{1}{2}}\Big\}.$$

Letting $n \to \infty$ we obtain

$$\int_0^{d(Au,z)} \psi(t)\,dt \leq (1-a) \int_0^{d(Au,z)} \psi(t)\,dt < \int_0^{d(Au,z)} \psi(t)\,dt,$$

a contradiction. Thus, $Au = z$.

Since the pair (A, S) is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$. If $Az \neq z$, using (9) we obtain

$$[1 + \alpha \int_0^{d(Sz,z)} \psi(t)\,dt] \int_0^{d(Az,z)} \psi(t)\,dt = [1 + \alpha \int_0^{d(Sz,Tv)} \psi(t)\,dt] \int_0^{d(Az,Bv)} \psi(t)\,dt$$
$$< \alpha[\int_0^{d(Az,Sz)} \psi(t)\,dt \cdot \int_0^{d(Bv,Tv)} \psi(t)\,dt + \int_0^{d(Az,Tv)} \psi(t)\,dt \cdot \int_0^{d(Sz,Bv)} \psi(t)\,dt]$$
$$+a \int_0^{d(Sz,Tv)} \psi(t)\,dt + (1-a)\, max\Big\{ \int_0^{d(Az,Sz)} \psi(t)\,dt,$$
$$\int_0^{d(Bv,Tv)} \psi(t)\,dt,\ (\int_0^{d(Az,Sz)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Az,Tv)} \psi(t)\,dt)^{\frac{1}{2}},$$
$$(\int_0^{d(Sz,Bv)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Az,Tv)} \psi(t)\,dt)^{\frac{1}{2}}\Big\}$$
$$= \int_0^{d(Az,z)} \psi(t)\,dt,$$

which is a contradiction. Thus, $Az = Sz = z$. Similarly, we can prove that $Bz = Tz = z$.

If (b) holds, then a point $z \in A(X) \cap T(X)$ is a weak tangent point to (A, T) and so there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Ty_n = z$. Because the pair of mapping (S, B) is tangential w.r.t the pair (A, T), we have

$$lim_{n\to\infty} Sx_n = lim_{n\to\infty} By_n = z.$$

Again, since $z \in A(X) \cap T(X)$, $z = Au = Tv$ for some $u, v \in X$. If $Bv \neq z$, using (9) we get

$$\begin{aligned}
&[1+\alpha \int_0^{d(Sx_n,Tv)} \psi(t)\,dt] \int_0^{d(Ax_n,Bv)} \psi(t)\,dt \;< \alpha[\int_0^{d(Ax_n,Sx_n)} \psi(t)\,dt \\
&\cdot \int_0^{d(Bv,Tv)} \psi(t)\,dt + \int_0^{d(Ax_n,Tv)} \psi(t)\,dt \cdot \int_0^{d(Sx_n,Bv)} \psi(t)\,dt] \\
&+a \int_0^{d(Sx_n,Tv)} \psi(t)\,dt + (1-a)\, max\Big\{ \int_0^{d(Ax_n,Sx_n)} \psi(t)\,dt, \\
&\int_0^{d(Bv,Tv)} \psi(t)\,dt,\; (\int_0^{d(Ax_n,Sx_n)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax_n,Tv)} \psi(t)\,dt)^{\frac{1}{2}}, \\
&(\int_0^{d(Sx_n,Bv)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax_n,Tv)} \psi(t)\,dt)^{\frac{1}{2}}\Big\}.
\end{aligned}$$

Letting $n \to \infty$ we obtain

$$\int_0^{d(z,Bv)} \psi(t)\,dt \leq (1-a) \int_0^{d(z,Bv)} \psi(t)\,dt < \int_0^{d(z,Bv)} \psi(t)\,dt,$$

which is a contradiction. Thus, $Bv = z$. Since the pair (B, T) is weakly compatible, we have $TBv = BTv$; i.e., $Bz = Tz$. If $Bz \neq z$, using (9) we obtain

$$\begin{aligned}
&[1+\alpha \int_0^{d(Sx_n,Tz)} \psi(t)\,dt] \int_0^{d(Ax_n,Bz)} \psi(t)\,dt \\
&< \alpha[\int_0^{d(Ax_n,Sx_n)} \psi(t)\,dt \cdot \int_0^{d(Bz,Tz)} \psi(t)\,dt + \int_0^{d(Ax_n,Tz)} \psi(t)\,dt \cdot \int_0^{d(Sx_n,Bz)} \psi(t)\,dt] \\
&+a \int_0^{d(Sx_n,Tz)} \psi(t)\,dt + (1-a)\, max\Big\{ \int_0^{d(Ax_n,Sx_n)} \psi(t)\,dt, \\
&\int_0^{d(Bz,Tz)} \psi(t)\,dt,\; (\int_0^{d(Ax_n,Sx_n)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax_n,Tz)} \psi(t)\,dt)^{\frac{1}{2}}, \\
&(\int_0^{d(Sx_n,Bz)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax_n,Tz)} \psi(t)\,dt)^{\frac{1}{2}}\Big\}
\end{aligned}$$

i.e.,

$$\begin{aligned}
\int_0^{d(z,Bz)} \psi(t)\,dt &< a \int_0^{d(z,Bz)} \psi(t)\,dt + (1-a) \int_0^{d(z,Bz)} \psi(t)\,dt \\
&= \int_0^{d(z,Bz)} \psi(t)\,dt,
\end{aligned}$$

which is a contradiction. Thus, $Bz = Tz = z$. Similarly, we can prove that $Az = Sz = z$.

If (c) holds, then we can draw the same conclusion as above. Finally, the uniqueness of z follows easily from (9). This completes the proof.

By setting $\alpha = 0$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying

$$\int_0^{d(Ax,By)} \psi(t)\,dt < a \int_0^{d(Sx,Ty)} \psi(t)\,dt + (1-a)\,max\Big\{ \int_0^{d(Ax,Sx)} \psi(t)\,dt,$$
$$\int_0^{d(By,Ty)} \psi(t)\,dt,\ (\int_0^{d(Ax,Sx)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax,Ty)} \psi(t)\,dt)^{\frac{1}{2}},$$
$$(\int_0^{d(Sx,By)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax,Ty)} \psi(t)\,dt)^{\frac{1}{2}}\Big\} \tag{10}$$

for all x, y in X for which the right-hand side of (10) is positive, where $0 < a < 1$ and ψ satisfies (6). Suppose that one of the conditions (a)-(c) of Theorem 2.5 holds; and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

If $\alpha = 0$, $B = A$ and $T = S$ in Theorem 2.5, we get the following corollary.

Corollary 2.7. Let A and S be mappings from a metric space (X, d) into itself satisfying

$$\int_0^{d(Ax,Ay)} \psi(t)\,dt < a \int_0^{d(Sx,Sy)} \psi(t)\,dt + (1-a)\,max\Big\{ \int_0^{d(Ax,Sx)} \psi(t)\,dt,$$
$$\int_0^{d(Ay,Sy)} \psi(t)\,dt,\ (\int_0^{d(Ax,Sx)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax,Sy)} \psi(t)\,dt)^{\frac{1}{2}},$$
$$(\int_0^{d(Sx,Ay)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax,Sy)} \psi(t)\,dt)^{\frac{1}{2}}\Big\} \tag{11}$$

for all x, y in X for which the right-hand side of (11) is positive, where $0 < a < 1$ and ψ satisfies (6). Suppose that there is a weak tangent $z \in A(X) \cap S(X)$ to (A, S) and the pair (A, S) is weak compatible. Then A and S have a unique common fixed point in X.

If $\psi(t) = 1$ in Theorem 2.5, we get the following corollary.

Corollary 2.8. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying

$$[1 + \alpha d(Sx, Ty)]d(Ax, By) < \alpha[d(Ax, Sx) \cdot d(By, Ty) + d(Ax, Ty) \cdot d(Sx, By)]$$

$$+ a\, d(Sx, Ty) + (1-a)\, max\Big\{ d(Ax, Sx),$$

$$d(By, Ty),\ (d(Ax, Sx))^{\frac{1}{2}} \cdot (d(Ax, Ty))^{\frac{1}{2}},$$

$$(d(Sx, By))^{\frac{1}{2}} \cdot (d(Ax, Ty))^{\frac{1}{2}} \Big\} \tag{12}$$

for all x, y in X for which the right-hand side of (12) is positive, where $0 < a < 1$, $\alpha \geq 0$. Suppose that one of the conditions (a)-(c) of Theorem 2.5 holds; and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

Corollary 2.9. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying

$$d(Ax, By) < a\, d(Sx, Ty) + (1-a)\, max\Big\{ d(Ax, Sx),$$

$$d(By, Ty),\ (d(Ax, Sx))^{\frac{1}{2}} \cdot (d(Ax, Ty))^{\frac{1}{2}},$$

$$(d(Sx, By))^{\frac{1}{2}} \cdot (d(Ax, Ty))^{\frac{1}{2}} \Big\} \tag{13}$$

for all x, y in X for which the right-hand side of (13) is positive, where $0 < a < 1$. Suppose that one of the conditions (a)-(c) of Theorem 2.5 holds and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

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