Certain Results on Differential Subordination for Some Classes of Multivalently Analytic Functions Associated with a Convolution Structure

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Abstract

Appealing to the familiar convolution structure of analytic functions, we introduce a general class of multivalently analytic functions, and derive several properties and characteristics of this function class by applying the differential subordination techniques. Relevant connections of the results stated here with those obtained in earlier works are also pointed out.

1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk

$$\mathbb{U} = \{z;\ z \in \mathbb{C} : |z| < 1\}.$$

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For the functions f and g in $\mathcal{A}_p$, we say that f is subordinate to g in $\mathbb{U}$, and write $f \prec g$ if there exists a function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < 1$ and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$. If f is univalent in $\mathbb{U}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $f \in \mathcal{A}_p$ be given by (1.1) and $g \in \mathcal{A}_p$ be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \qquad (p \in \mathbb{N}), \tag{1.2}$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k := (g * f)(z). \tag{1.3}$$

For a given function $g(z) \in \mathcal{A}_p$ (defined by (1.2)), we define here a new class $\mathcal{J}_p(g; \alpha, A, B)$ of functions belonging to the subclass of $\mathcal{A}_p$ which consist of functions $f(z)$ of the form (1.1) satisfying the following subordination:

$$(1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{((f*g)(z))'}{z^{p-1}} \prec \frac{1+Az}{1+Bz} \tag{1.4}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ \alpha > 0\,;\ -1 \leq B < A \leq 1).$$

We put

$$\mathcal{J}_p\left(g; \alpha, 1 - \frac{2\beta}{p}, -1\right) = \mathcal{J}_p^*(g; \alpha, \beta), \tag{1.5}$$

where $\mathcal{J}_p^*(g; \alpha, \beta)$ denotes the class of functions $f \in \mathcal{A}_p$ satisfying the following inequality:

$$\Re\left((1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{((f*g)(z))'}{z^{p-1}}\right) > \frac{\beta}{p}. \tag{1.6}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ \alpha > 0\,;\ 0 \leq \beta < p)$$

It may be observed that several new and known interesting subclasses are deducible from our function class $\mathcal{J}_p(g; \alpha, A, B)$. We mention below some of the interesting subclasses of the class defined above by (1.4).

If the coefficients b_k in (1.2) and the value of α in (1.4) are, respectively, chosen as follows:

$$b_k = \frac{(\alpha_1)_{k-p} \cdots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}\,(k-p)!} \qquad \text{and} \qquad \alpha = \frac{p\lambda}{\alpha_i} \tag{1.7}$$

$$(\alpha_i > 0\ (i = 1, \ldots, q);\ \beta_j > 0\ (j = 1, \ldots, s);\ \lambda > 0;\ q \leq s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

and in the process making use of the identity:

$$z\left(H_s^q[\alpha_i]f(z)\right)' = \alpha_i\left(H_s^q[\alpha_i+1]f(z)\right) - (\alpha_i - p)\left(H_s^q[\alpha_i]f(z)\right) \qquad (i = 1, \ldots, q) \tag{1.8}$$

in (1.4), then the class $\mathcal{J}_p(g;\alpha,A,B)$ reduces to the function class studied very recently by Liu [7].

It may be noted here that the operator

$$\left(H_s^q[\alpha_1]f\right)(z) := H_s^q(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)$$

involved in the identity (1.8) is the Dziok-Srivastava linear operator (see, for details,[5]; see also [6]) which contains such well known operators as the Hohlov linear operator, Saitoh generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized version, the Barnardi-Libera-Livingston operator, and the Srivastave-Owa fractional derivative operator. One may refer to the papers [5] and [6] for further details and references for these operators. The Dziok- Srivastava linear operator defined in [5] has further been generalized by Dziok and Raina [3] (see also [4]).

On the other hand, if we set the coefficients b_k in (1.2) and the value of the parameter α in (1.4), respectively, as follows:

$$b_k=\left(\frac{p+1}{k+1}\right)^{\sigma}\quad(\sigma>0;\ k\geq p+1\ (p\in\mathbb{N}))\quad\text{and}\quad\alpha=\frac{\lambda}{(p+1)}\tag{1.9}$$

and applying the identity:

$$z\left(I^{\sigma}f(z)\right)'=(p+1)I^{\sigma-1}f(z)-I^{\sigma}f(z)\qquad(p\in\mathbb{N};\ \sigma>0)\tag{1.10}$$

in (1.4), then the class $\mathcal{J}_p(g;\alpha,A,B)$ reduces to the class $\Omega_p^{\sigma}(A,B,\lambda)$ studied very recently by Sham *et al.* [13](see also [11]), where the operator $I^{\sigma}f(z)$ is defined by ([13, p. 1, Eq. (1.1)])

$$\begin{aligned}I^{\sigma}f(z)&=\frac{(p+1)^{\sigma}}{z\,\Gamma(\sigma)}\int_0^z\left(log\ \frac{z}{t}\right)^{\sigma-1}f(t)dt\\&=z^p+\sum_{k=p+1}^{\infty}\left(\frac{p+1}{k+1}\right)^{\sigma}a_kz^k\qquad(f\in\mathcal{A}_p;\ p\in\mathbb{N};\ \sigma>0).\end{aligned}$$

Moreover, if we choose the coefficients b_k in (1.2) and the value of the parameter α in (1.4), respectively, as follows:

$$b_k=\frac{k}{p}\left(\frac{k+\mu}{p+\mu}\right)^{r}\quad(p\in\mathbb{N};\ r\in\mathbb{N}_0;\ \mu\geq0)\quad\text{and}\quad\alpha=1,\tag{1.11}$$

then the class $\mathcal{J}_p(g;\alpha,A,B)$ transforms into a (presumably) new class $\mathcal{R}_p^r(\mu,A,B)$ defined by

$$\mathcal{R}_p^r(\mu,A,B):=\left\{f:\ \ f\in\mathcal{A}_p\ \ \text{and}\ \ \frac{1}{p}\frac{[I_p(r,\mu)f]'(z)}{z^{p-1}}\prec\frac{1+Az}{1+Bz}\right\}\tag{1.12}$$

$$(z\in\mathbb{U};\ p\in\mathbb{N};\ r\in\mathbb{N}_0;\ \mu\geq0;\ -1\leq B<A\leq1)$$

involving the operator $I_p(r,\mu)$ which is given by ([16])

$$I_p(r,\mu)f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{k+\mu}{p+\mu}\right)^{r}a_kz^k\tag{1.13}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ r \in \mathbb{N}_0;\ \mu \geq 0).$$

The class $\mathcal{R}_p^r(1, A, B)$ was earlier studied by Srivastava *et al.* [14]. Furthermore, on suitably specializing the coefficients b_k in (1.2) and the parameter α in (1.4) and (1.6), one may obtain other function classes from the classes $\mathcal{J}_p(g; \alpha, A, B)$ and $\mathcal{J}_p^*(g; \alpha, \beta)$ investigated among others by Dinggong and Liu [2] and Srivastava *et al.* [15].

In the present paper we obtain various useful properties and characteristics of the function class $\mathcal{J}_p(g; \alpha, A, B)$ and $\mathcal{J}_p^*(g; \alpha, \beta)$ (defined above) by using the techniques involving the Briot-Bouquet differential subordination. Several corollaries are deduced from the main results and their connections with known results are also pointed out.

2 Preliminaries and Key Lemmas

In order to derive our main results, we recall here the following lemmas:

Lemma 1 ([8, p. 132]). *Let $q(z)$ be analytic and univalent in $\mathbb{U}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) = zq'(z)\ \phi[q(z)], \qquad h(z) = \theta[q(z)] + Q(z)$$

and suppose that

(i) $Q(z)$ is univalent and starlike in $\mathbb{U}$

(ii) $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) = \Re\left(\dfrac{\theta'[q(z)]}{\phi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right) > 0 \qquad (z \in \mathbb{U}).$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$, and

$$\theta[p(z)] + zp'(z)\ \phi[p(z)] \prec \theta[q(z)] + zq'(z)\ \phi[q(z)] = h(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominent.

On choosing $\theta(w) := \psi w$ and $\phi(w) := \gamma$ in Lemma 1, we obtain

Lemma 2 ([13]). *Let $q(z)$ be convex univalent function in $\mathbb{U}$ with $q(0) = 1$ and $\psi,\ \gamma \in \mathbb{C}$. Further, assume that*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left(0, -\frac{\psi}{\gamma}\right) \qquad (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$, and

$$\psi\ p(z) + \gamma z p'(z) \prec \psi\ q(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominent.

Lemma 3 ([9, p.2, Lemma 1.1]). *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and γ be a complex number satisfying $\Re(\gamma) \geq 0$ $(\gamma \neq 0)$, then*

$$\Re[p(z) + \gamma z p'(z)] > \beta \qquad (0 \leq \beta < 1)$$

implies that

$$\Re\,(p(z)) > \beta + (1-\beta)(2\lambda - 1),$$

where λ is given by

$$\lambda = \lambda_{\,\Re(\gamma)} = \int_0^1 \left(1 + t^{\,\Re(\gamma)}\right)^{-1}\, dt.$$

Lemma 4 ([10], see also [15, p. 325, Lemma 5]). *Let $\phi(z)$ be analytic in $\mathbb{U}$ with*

$$\phi(0) = 1 \qquad \text{and} \qquad \phi(z) \neq 0 \qquad (0 < |z| < 1).$$

Also, let

$$\left|\frac{\nu(A-B)}{B} - 1\right| \le 1 \qquad (-1 \le B < A \le 1;\ B \neq 0;\ \nu \in \mathbb{C}\backslash\{0\})$$

or

$$\left|\frac{\nu(A-B)}{B} + 1\right| \le 1 \qquad (-1 \le B < A \le 1;\ B \neq 0;\ \nu \in \mathbb{C}\backslash\{0\}).$$

If $\phi(z)$ satisfies the following subordination condition:

$$1 + \frac{z\,\phi'(z)}{\nu\,\phi(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

then

$$\phi(z) \prec \psi(z) := (1+Bz)^{\frac{\nu(A-B)}{B}} \qquad (z \in \mathbb{U})$$

and $\psi(z)$ is the best dominent.

3 Main Results

Our first main result is given by Theorem 1 below.

Theorem 1. *Let $q(z)$ be univalent in $\mathbb{U}$ and $q(z)$ satisfy the inequality*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0 \qquad (z \in \mathbb{U}). \tag{3.1}$$

If $f \in \mathcal{A}_p$ satisfies the following subordination:

$$(1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{((f*g)(z))'}{z^{p-1}} \prec q(z) + \frac{\alpha}{p} zq'(z) \qquad (z \in \mathbb{U};\ \alpha > 0;\ p \in \mathbb{N}), \tag{3.2}$$

then

$$\frac{(f*g)(z)}{z^p} \prec q(z)$$

and $q(z)$ is the best dominent.

Proof. Define a function $\Theta(z)$ by

$$\Theta(z) = \frac{(f*g)(z)}{z^p}, \tag{3.3}$$

which is of the form

$$\Theta(z) = 1 + c_1 z + c_2 z^2 + ... \tag{3.4}$$

and is analytic in $\mathbb{U}$. Differentiating both sides of (3.3), we get

$$\frac{((f * g)(z))'}{z^{p-1}} = p\,\Theta(z) + z\,\Theta'(z). \tag{3.5}$$

Using (3.2), (3.3) and (3.5), the subordination (3.2) then becomes

$$\Theta(z) + \frac{\alpha}{p}\, z\Theta'(z) \prec q(z) + \frac{\alpha}{p}\, zq'(z).$$

Now appealing to Lemma 2 when ($\gamma = \frac{\alpha}{p}$ and $\psi = 1$), we conclude that $\Theta(z) \prec q(z)$. This complete the proof of Theorem 1.

Upon setting

$$q(z) = \frac{1 + Az}{1 + Bz} \qquad (-1 \leq B < A \leq 1),$$

Theorem 1 yields the following result.

Corollary 1. *Let* $-1 \leq B < A \leq 1$ *and*

$$\Re\left(\frac{1 - Bz}{1 + Bz}\right) + \frac{p}{\alpha} > 0 \qquad (z \in \mathbb{U};\ 0 \leq \alpha < p;\ p \in \mathbb{N}).$$

If $f \in \mathcal{A}_p$ *satisfies the following subordination:*

$$(1 - \alpha)\,\frac{(f * g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{((f * g)(z))'}{z^{p-1}} \prec \frac{\alpha(A - B)z}{p(1 + Bz)^2} + \frac{1 + Az}{1 + Bz} \qquad (z \in \mathbb{U}),$$

then

$$\frac{(f * g)(z)}{z^p} \prec \frac{1 + Az}{1 + Bz}.$$

If we choose

$$p = 1,\ \ B = 0,\ \ A = \frac{1}{1 + \alpha} \ \text{ and } \ g(z) = \frac{z}{(1 - z)^2},$$

then Corollary 1 immediately yields the following result.

Corollary 2. *If* $f(z) \in \mathcal{A}$ $(\mathcal{A} := \mathcal{A}_1)$ *satisfies*

$$f'(z) + \alpha\, zf''(z) \prec 1 + z \qquad (z \in \mathbb{U};\ \alpha \geq 0),$$

then

$$f'(z) \prec 1 + \frac{1}{1 + \alpha}\, z.$$

Theorem 2. *If* $f(z) \in \mathcal{J}_p^*(g;\alpha,p\beta)$, *then*

$$\Re\left(\frac{(f*g)(z)}{z^p}\right) > \beta + (1-\beta)(2\lambda - 1) \quad (z \in \mathbb{U};\ 0 \le \beta < 1), \tag{3.6}$$

where

$$\lambda = \int_0^1 \left(1 + t^{\frac{\alpha}{p}}\right)^{-1} dt \qquad (0 \le \alpha < p;\ p \in \mathbb{N}).$$

Proof. Let $f(z) \in \mathcal{J}_p^*(g;\alpha,p\beta)$. Following the proof of Theorem 1, we find that

$$\begin{aligned} &\Re\left((1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{((f*g)(z))'}{z^{p-1}}\right) \\ &= \Re\left(\Theta(z) + \frac{\alpha}{p}\,z\,\Theta'(z)\right) \\ &> \beta \qquad (0 \le \beta < 1). \end{aligned}$$

The application of Lemma 3 with $\gamma = \frac{\alpha}{p}$ eventually leads us to the desired assertion (3.6) of Theorem 2, and this complete the proof of Theorem 2.

If we set

$$\alpha = p = 1 \qquad \text{and} \qquad g(z) = \frac{z}{1-z}$$

in Theorem 2, then we obtain the following result.

Corollary 3. *Let* $f(z) \in \mathcal{A}$ *satisfy the inequality*

$$\Re(f'(z)) > \beta \qquad (0 \le \beta < 1),$$

then

$$\Re\left(\frac{f(z)}{z}\right) > 2\beta + 2(1-\beta)\ ln2 - 1.$$

Similarly, on setting

$$\alpha = \frac{1}{2}, \qquad p = 1 \qquad \text{and} \qquad g(z) = \frac{z}{(1-z)^2}$$

in Theorem 2, we obtain the following result.

Corollary 4. *Let* $f(z) \in \mathcal{A}$ *satisfy the inequality*

$$\Re\left(f'(z) + \frac{1}{2}\,zf''(z)\right) > \beta \qquad (0 \le \beta < 1),$$

then

$$\Re(f'(z)) > 3 - 2\beta - 4(1-\beta)\ ln2.$$

Theorem 3. *Let*

$$f \in \mathcal{A}_p; \quad 0 \le \alpha < p; \quad -1 \le a < 0 \quad \text{and} \quad -1 < A < 1.$$

Suppose also that

$$\frac{(f*g)(z)}{z^p} \neq 0 \qquad (z \in \mathbb{U})$$

and

$$\left(\frac{(f*g)(z)}{z^p}\right)^a \left((1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{((f*g)(z))'}{z^{p-1}}\right) \prec h(z) \quad (z \in \mathbb{U}), \quad (3.7)$$

where

$$h(z) = \left(\frac{1+Az}{1-z}\right)^a \left(\frac{1+Az}{1-z} + \frac{p}{\alpha}\frac{(A+1)z}{(1-z)^2}\right).$$

Then

$$\frac{(f*g)(z)}{z^p} \prec \frac{1+Az}{1-z}.$$

Proof. Define a function $\Lambda(z)$ by

$$\Lambda(z) = \frac{(f*g)(z)}{z^p} \qquad (z \in \mathbb{U}) \tag{3.8}$$

which is of the form (3.4), and hence analytic in $\mathbb{U}$ with $\Lambda(0) = 1$. Differencing (3.8), we find that the subordination relation (3.7) becomes

$$[\Lambda(z)]^{a+1} + \frac{\alpha}{p}[\Lambda(z)]^a\, z\Lambda'(z) \prec h(z). \tag{3.9}$$

Setting

$$q(z) = \frac{1+Az}{1-z}, \qquad \theta(w) = w^{a+1} \qquad \text{and} \qquad \phi(w) = \frac{p}{\alpha}w^a \tag{3.10}$$

in Lemma 1, wherein, we note that $q(z)$ is analytic and univalent in $\mathbb{U}$ with $q(0) = 1$, and both $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{U}$ with $\phi(w) \neq 0$ in $C\backslash\{0\}$. We observe that

$$Q(z) = zq'(z)\,\phi[q(z)] = \frac{\alpha(1+A)z(1+Az)^a}{p(1-z)^{a+2}} \tag{3.11}$$

is univalent and starlike in $\mathbb{U}$ which follows by inferring that

$$\begin{aligned}\Re\left(\frac{zQ'(z)}{Q(z)}\right) &= \Re\left[1 + a\frac{Az}{1+Az} + (a+2)\frac{z}{1-z}\right] > 1 - \frac{a|A|}{1+|A|} - \frac{a+2}{2}\\ &= \frac{-a(1+3|A|)}{2(1+|A|)} > 0.\end{aligned}$$

Further, we infer that

$$h(z) = \theta[q(z)] + Q(z) = \left(\frac{1+Az}{1-z}\right)^a \left(\frac{1+Az}{1-z} + \frac{\alpha(1+A)z}{p(1-z)^2}\right) \qquad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right) =$$
$$\frac{p(a+1)}{\alpha} + \Re\left(\frac{zQ'(z)}{Q(z)}\right) \geq 0 \qquad (z \in \mathbb{U};\ \alpha \neq 0) \quad (3.12)$$

for $z \in \mathbb{U}$. The inequality (3.12) shows that the functions $h(z)$ is close-to-convex and univalent in $\mathbb{U}$. Hence, it follows from (3.9) to (3.12) that

$$\theta[\Lambda(z)] + z\,\Lambda'(z)\,\phi[\Lambda(z)] \prec \theta[q(z)] + zq'(z)\,\phi[q(z)] = h(z).$$

Therefore, by virtue of Lemma 1, we conclude that $\Lambda(z) \prec q(z)$ which completes the proof of Theorem 3.

Let us put

$$h(z) = 1 + \frac{\alpha}{1-z} - \frac{\alpha}{1+Az}, \tag{3.13}$$

which assumes real values of z with $h(0) = 1$, and $h(\mathbb{U})$ is symmetric with respect to the real axis, and

$$\Re\{h(z)\} > 1 + \frac{\alpha}{2} - \frac{\alpha}{1-|A|} \qquad (z \in \mathbb{U}).$$

Consequently, by letting

$$a = -1 \qquad \text{and} \qquad \alpha = 1$$

in Theorem 3, we obtain the following corollary.

Corollary 5. *Let*

$$-1 < A < 1 \qquad \textit{and} \qquad \frac{(f*g)(z)}{z^p} \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}_p$ satisfies the inequality:

$$\Re\left(\frac{z\,((f*g)(z))'}{(f*g)(z)}\right) > \frac{3}{2}p - \frac{p}{1-|A|},$$

then

$$\frac{(f*g)(z)}{z^p} \prec \frac{1+Az}{1-z}.$$

Further, on setting

$$p = 1, \qquad A = 0 \qquad \text{and} \qquad g(z) = \frac{z}{1-z}$$

in Corollary 5, we get

Corollary 6. *Let*

$$\frac{f(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ satisfies the inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2},$$

then

$$\frac{f(z)}{z} \prec \frac{1}{1-z}.$$

Remark 1. We observe that if we use the parametric substitutions given by (1.7) with $p = 1$, apply the identity (1.8) for $p = 1$, then Theorem 3 corresponds to the result given recently by Aghalary *et al.* [1, p. 535, Theorem 2.2].

Theorem 4. *Let $0 \leq \rho < 1$ and γ be a complex number with $\gamma \neq 0$ such that either*

$$\left|\frac{2\gamma(1-\rho)p}{\alpha} - 1\right| \leq 1 \quad \text{or} \quad \left|\frac{2\gamma(1-\rho)p}{\alpha} + 1\right| \leq 1.$$

If $f(z) \in \mathcal{A}_p$ satisfies the inequality:

$$\Re\left(\frac{z\,((f*g)(z))'}{(f*g)}\right) > \frac{p(\rho+\alpha-1)}{\alpha} \qquad (z \in \mathbb{U}), \tag{3.14}$$

then

$$\left(\frac{(f*g)(z)}{z^p}\right)^{\gamma} \prec (1-z)^{-\frac{2\gamma(1-\rho)p}{\alpha}} = q(z), \tag{3.15}$$

where $q(z)$ is the best dominent.

Proof. Let $f(z) \in \mathcal{A}_p$ and assume that

$$\Omega(z) = \left(\frac{(f*g)(z)}{z^p}\right)^{\gamma} \qquad (z \in \mathbb{U}). \tag{3.16}$$

Differentiating (3.16), we get

$$\frac{z\,\Omega'(z)}{\Omega(z)} = \gamma\left(\frac{z\,((f*g)(z))'}{(f*g)(z)} - p\right). \tag{3.17}$$

Applying now (3.16) and (3.17) in (3.14), we obtain

$$\begin{aligned}\frac{\alpha}{p}\,\frac{z\,((f*g)(z))'}{(f*g)(z)} - \alpha + 1 &= 1 + \frac{\alpha}{\gamma p}\frac{z\,\Omega'(z)}{\Omega(z)} \\ &\prec \frac{1+(1-2\rho)z}{1-z}.\end{aligned}$$

Using Lemma 4 when $\nu = \frac{\gamma p}{\alpha}$, we conclude that $\Omega(z) \prec q(z)$, thereby, proving Theorem 4.

An alternative proof of Theorem 4 can also be worked out by using Lemma 1. Indeed, in this case, when we set

$$q(z) = (1-z)^{-\frac{2\gamma(1-\rho)p}{\alpha}}, \qquad \theta(w) = 1 \qquad \text{and} \qquad \phi(w) = \frac{\alpha}{p\gamma w}$$

in Lemma 1 and apply similar technique of Theorem 3, we arrive at the desired result.

Corollary 7. *Let $0 \leq \rho < 1$ and γ $(\gamma \geq 1)$ be a real number. If $f(z) \in \mathcal{A}_p$ satisfies the inequality (3.14), then*

$$\Re\left(\frac{(f*g)(z)}{z^p}\right)^{\frac{\alpha}{2\gamma(1-\rho)p}} > 2^{-\frac{1}{\gamma}} \qquad (z \in \mathbb{U}). \tag{3.18}$$

The bound $2^{-\frac{1}{\gamma}}$ is the best possible.

Proof. Let there exist a Schwarz function $w(z)$ which is analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}).$$

The subordination relation (3.15) can be expressed as

$$\left(\frac{(f*g)(z)}{z^p}\right)^{\gamma} = (1-w(z))^{-\frac{2\gamma(1-\rho)p}{\alpha}},$$

which implies that

$$\left(\frac{(f*g)(z)}{z^p}\right)^{\frac{\gamma\alpha}{2(1-\rho)p}} = (1-w(z))^{-\gamma}. \tag{3.19}$$

By considering the real parts on both sides of (3.19) and applying the elementary inequality:

$$\Re(w^{\frac{1}{m}}) \geq [\Re(w)]^{\frac{1}{m}} \quad \text{for} \quad \Re(w) > 0 \quad \text{and} \quad m \geq 1,$$

we get (3.18). This completes the proof of Corollary 7.

Remark 2. We observe that if we use the parametric substitutions given by (1.7) with $\lambda = 1$ and apply the identity (1.8), then Theorem 4 and Corollary 7 correspond to the results given recently by Liu [7, p. 4, Theorem 2.5; p.5, Corollary 2.6]. Also, making use of the parametric substitutions given by (1.9) with $\lambda = p$ and applying the identity(1.10), then Theorem 4 and Corollary 7 yield the *corrected* version of recently established results due to Ozkan [11, p.6, Theorem 2.6; p.7, Corollary 2.7].

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