On the positive solutions of certain semi-linear elliptic equations

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Abstract

We establish that the elliptic equation $\Delta u + f(x,u) + g(|x|)x \cdot \nabla u = 0$, where $x \in \mathbb{R}^n$, $n \geq 3$, and $|x| > A > 0$, has a positive solution which decays to 0 as $|x| \to +\infty$ under mild restrictions on the functions f, g. The main theorem improves substantially upon the conclusions of the recent paper [M. Ehrnström, Positive solutions for second-order nonlinear differential equations, Nonlinear Anal. TMA 64 (2006), 1608–1620]. Its proof relies on a sharp result of non-oscillation of linear ordinary differential equations and on the comparison method.

1 Introduction

This note, motivated by the recent papers [13, 2, 14], is concerned with the existence of a positive solution to the linear differential equation

$$u'' + a(t)u = 0, \qquad t \geq t_0 \geq 1, \tag{1}$$

such that

$$u'(t) - \frac{u(t)}{t} \leq 0, \qquad u(t) = o(t) \qquad \text{as } t \to +\infty. \tag{2}$$

Here, the functional coefficient $a : [t_0, +\infty) \to [0, +\infty)$ is taken continuous and with (eventual) isolated zeros. We tacitly assume that, given the large literature

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regarding the equation (1), of interest would be only to get (nearly) optimal hypotheses for the existence of such a solution.

The preoccupation in studying the problem (1), (2) comes from an investigation of the existence of positive, vanishing at $+\infty$, solutions to the semi-linear elliptic equation of second order

$$\Delta u + f(x,u) + g(|x|)x \cdot \nabla u = 0, \qquad x \in G_A, \tag{3}$$

where $G_A = \{x \in \mathbb{R}^n : |x| > A\}$ and $n \geq 3$, that relies heavily upon (sharp) criteria regarding the long-time behavior of various solutions to ordinary differential equations. Making a presentation of recent literature about positive solutions to elliptic partial differential equations and their behavior when x reaches the spatial infinity is an almost impossible task, so we prefer to cite only a few studies [1, 2, 6, 7, 9, 10], [12] – [14], [18, 19], [21] – [24] which contain methods in the same spirit as ours.

Following [6, 21, 22, 24], we consider that the functions $f : G_A \times \mathbb{R} \to \mathbb{R}$ and $g : [A, +\infty) \to \mathbb{R}$ are locally Hölder continuous. Moreover,

$$0 \leq f(x,U) \leq m(|x|)U, \qquad x \in G_A,\ U \in [0,\zeta], \tag{4}$$

for some $\zeta > 0$ and the continuous application $m : [A, +\infty) \to [0, +\infty)$. The regularity assumptions upon f, g are sufficient for applying the comparison method [16] to the analysis of (3). In fact, given u a positive solution of (1), we would like the function

$$U(x) = U(|x|) = \frac{u(t)}{t}, \qquad \text{where } |x| = \beta(t) = \left(\frac{t}{n-2}\right)^{\frac{1}{n-2}} \tag{5}$$

and $t \geq t_0 > \max\{1, (n-2)A^{n-2}\}$, to be a super-solution of (3) satisfying the additional restriction

$$x \cdot \nabla U(x) \leq 0, \qquad x \in G_B, \tag{6}$$

for some $B > A$. We take herein

$$a(t) = \frac{1}{(n-2)^2} \frac{\beta'(t)}{[\beta(t)]^{n-3}} \cdot m(\beta(t)), \qquad t \geq t_0. \tag{7}$$

The reason for introducing (6), see [13], is that, when g takes only nonnegative values, this additional requirement for the positive solution U of the linear elliptic equation (of comparison)

$$\Delta U + m(|x|)U = 0, \qquad x \in G_B,$$

will allow for a complete removal of the conditions regarding g (given mostly by convergent improper integrals) from the hypotheses of various theorems in the recent literature. Further developments of this observation are given in [1, 2, 10, 14].

Condition (6), translated into the language of ordinary differential equations, reads as

$$u'(t) - \frac{u(t)}{t} \leq 0 \qquad \text{for every sufficiently large } t. \tag{8}$$

Here, by means of an integral Riccati equation associated with (1), see also [1], we establish the existence of a solution to (1), (2) in nearly optimal circumstances. The result is used to derive a criterion for the existence of positive solutions to (3) that decay to 0 as $|x| \to +\infty$. It improves consistently upon the conclusions of [1, 2, 10, 14].

2 Positive solution to the problem (1), (2)

Assume that the differential equation (1) has already a positive solution $u(t)$. Then, the classical Hartman non-oscillation theorem [17] implies that

$$\int_{t_0}^{+\infty} a(t)dt < +\infty. \tag{9}$$

Further, since $u''(t) \leq 0$ in $[t_0, +\infty)$, there exists $\lim_{t\to+\infty} u'(t) = l \in [0, u'(t_0)]$. Also, we have

$$u'(t) = l + \int_t^{+\infty} a(s)u(s)ds, \qquad t \geq t_0. \tag{10}$$

If $l > 0$ then, via L'Hôpital's rule, we deduce from (10) that

$$u(t) = l \cdot t + o(t) \qquad \text{for } t \to +\infty$$

and

$$\int_{t_0}^{+\infty} ta(t)dt < +\infty. \tag{11}$$

The condition (11) has been already discussed in the literature [6, 7, 9], [12] – [14], [24]. In fact, the condition is necessary and sufficient for the existence of bounded positive solutions of (1) that verify (8), see [13, 10].

We shall focus here on the case $l = 0$. According to (10), the presumable positive solution u verifies the inequality

$$u'(t) \geq \left(t \int_t^{+\infty} a(s)ds \right) \frac{u(t)}{t}, \qquad t \geq t_0.$$

This means that, if u is a solution of the problem (1), (2), then the functional coefficient a must be confined by

$$t \int_t^{+\infty} a(s)ds \leq 1 \tag{12}$$

throughout $[t_0, +\infty)$. For various implications of (12), see the papers [4, 25].

The next result, in the spirit of Opial [8], provides a connection between sharp non-oscillation results for (1) and the existence of solutions obeying (2). Though its proof might look surprisingly simple, we have found that it is almost impossible to improve its main hypothesis (13).

Theorem 1. *Assume that (12) holds true and there exists a continuous function* $\lambda : [t_0, +\infty) \to [0, +\infty)$ *such that*

$$\int_t^{+\infty} \left[\int_s^{+\infty} a(\tau) d\tau + \lambda(s) \right]^2 ds \le \lambda(t), \qquad t \ge t_0. \tag{13}$$

Consider also that

$$\int_{t_0}^{+\infty} t a(t) dt = +\infty. \tag{14}$$

Then, the problem (1), (2) has a positive unbounded solution in $[t_0, +\infty)$.

Proof. We shall proceed in three steps.

The first step. Suppose that for a positive solution $u(t)$ of (1) there exists $t_1 \ge t_0$ such that $u'(t_1) < \frac{u(t_1)}{t_1}$. Then, the following holds true

$$u(t) - u(t_1) = \int_{t_1}^{t} u'(s) ds \ge (t - t_1) u'(t),$$

since the derivative u' is monotone non-increasing throughout $[t_1, +\infty)$ (recall that $u''(t) = -a(t)u(t) \le 0$), and respectively

$$\begin{aligned} u(t) &\ge t_1 \left[\frac{u(t_1)}{t_1} - u'(t) \right] + t u'(t) \\ &\ge t_1 \left[\frac{u(t_1)}{t_1} - u'(t_1) \right] + t u'(t) > t u'(t), \qquad t > t_1. \end{aligned}$$

We have, consequently, either $u'(t) \ge \frac{u(t)}{t}$ for all $t \ge t_0$ or $u'(t) < \frac{u(t)}{t}$ in $[t_1, +\infty)$ for a certain $t_1 \ge t_0$.

The second step. Introduce the set

$$S = \left\{ b \in (L^2 \cap L^\infty)((t_0, +\infty), \mathbb{R}) : \int_t^{+\infty} [b(s)]^2 ds \le \lambda(t) \text{ for all } t \ge t_0 \right\}.$$

The partial order $'' \le ''$, given by $b_1 \le b_2$ if and only if $b_1(t) \le b_2(t)$ almost everywhere in $(t_0, +\infty)$, makes $(S, \le)$ a complete lattice.

The operator $T : S \to S$ with the formula

$$T(b)(t) = \int_t^{+\infty} [b(s)]^2 ds + \int_t^{+\infty} a(s) ds, \qquad t \ge t_0, \tag{15}$$

is well defined by means of (13).

The application T is isotone, that is, $T(b_1) \le T(b_2)$ whenever $b_1 \le b_2$, and, the functional coefficient $a(t)$ being nonnegative, it satisfies the inequality $0 \le T(0)$. By application of the Knaster-Tarski fixed point theorem [11, p. 14], T has a fixed point in S, denoted b_0.

Since $b_0 \in L^\infty$, we deduce from (15) that $b_0 = T(b_0)$ is a locally Lipschitz function. Using again (15), we conclude that b_0 is continuously differentiable throughout $[t_0, +\infty)$. This means that the function $u_0(t) \equiv \exp\left(\int_{t_0}^{t} b_0(s)ds\right)$, where $t \geq t_0$, is a positive classical (C^2) solution of (1).

The third step. Suppose now, taking into account the discussion from the first step, that

$$u_0'(t) \geq \frac{u_0(t)}{t}, \qquad t \geq t_0.$$

This means that the mapping $t \mapsto \frac{u_0(t)}{t}$ is monotone non-decreasing throughout $[t_0, +\infty)$.

Accordingly, via (10) for $l = 0$, we deduce that

$$u_0'(t) = \int_t^{+\infty} a(s)u_0(s)ds \geq \int_t^{+\infty} sa(s)ds \cdot \frac{u_0(t)}{t} > 0,$$

contrary to (14). ■

Remark 1. In the paper [1], the function a from (1) is subjected to the Hille-type restriction

$$\int_t^{+\infty} a(s)ds \leq \frac{q(t)}{t} - \int_t^{+\infty} \left[\frac{q(s)}{s}\right]^2 ds, \qquad t \geq t_0, \tag{16}$$

where the function $q : [t_0, +\infty) \to (0, 1]$ is continuous and with $\sup_{t \geq t_0}\{q(t)\} < \frac{1}{2}$. The case where $q(t) = c \in \left(0, \frac{1}{2}\right]$ for all $t \geq t_0$ is discussed in [19]. The restriction (16) is very permissive, however, it is included in the circumstances described in our Theorem 1. In fact, by taking $\lambda(t) = \int_t^{+\infty} \left[\frac{q(s)}{s}\right]^2 ds$ for all $t \geq t_0$, the hypothesis (13) follows immediately from (16).

Remark 2. To give a hint about the sharpness of (13), let us notice that an integration by parts in formula (10) leads to

$$t \int_t^{+\infty} \left\{ \left[\int_s^{+\infty} a(\tau)d\tau\right]^2 + a(s) \right\} ds \leq 1, \qquad t \geq t_0,$$

see also the analysis in [5]. A technique for building very flexible functions λ can be derived from the methods in [3, 20].

3 Application to (3)

The following lemma will be needed in the investigation.

Lemma 1. (see [21]) *If, for some $B > A$, there exist a nonnegative subsolution w and a positive supersolution v to (3) in G_B, such that $w(x) \leq v(x)$ for $x \in \overline{G}_B$, then (3) has a solution u in G_B such that $w \leq u \leq v$ throughout $\overline{G}_B$. In particular, $u = v$ on $|x| = B$.*

Our main contribution here is the next result.

Theorem 2. *Assume that $g(\alpha) \geq 0$ for any $\alpha \geq A$. Suppose also that the following estimates*

$$\int_r^{+\infty} \tau^{n-3} \left[\lambda((n-2)\tau^{n-2}) + c_n \int_\tau^{+\infty} \frac{m(\xi)}{\xi^{n-3}} d\xi \right]^2 d\tau \leq c_n \cdot \lambda((n-2)r^{n-2}),$$

where $c_n = (n-2)^{-2}$,

$$\sqrt{c_n} \cdot r^{n-2} \int_r^{+\infty} \frac{m(\tau)}{\tau^{n-3}} d\tau \leq 1 \qquad \text{and} \qquad \int_A^{+\infty} rm(r)dr = +\infty$$

hold true in $[A, +\infty)$.

Then, the equation (3) has a positive solution $u(x)$, defined in G_B for some $B > A$, such that $\lim\limits_{|x| \to +\infty} u(x) = 0$.

Proof. Consider the positive, twice continuously differentiable functions given by

$$U(x) = y(r) = \frac{u_0(t)}{t}, \qquad t \geq t_1, \tag{17}$$

where $r = |x| = \beta(t)$. Here, u_0 is the solution of problem (1), (2) obtained at Theorem 1 and the number t_1 is given in the first step of the previous proof. Also, t_1 is eventually increased in order to have $y(\beta(t)) \leq \zeta$ throughout $[t_1, +\infty)$, see (4), and we put $B = \beta(t_1)$.

By a straightforward computation we get that

$$t\beta'(t) = \frac{1}{n-2}\beta(t) \tag{18}$$

and

$$\begin{cases} \frac{du_0}{dt} = y + t\beta'(t)\frac{dy}{dr} \\ \frac{d^2u_0}{dt^2} = \frac{n-1}{n-2}\beta'(t)\frac{dy}{dr} + \frac{\beta(t)\beta'(t)}{n-2}\frac{d^2y}{dr^2}. \end{cases} \tag{19}$$

Further, taking into account (18), (19), we have

$$\begin{aligned} &r^{n-1}\left(\Delta U + f(x,U) + g(|x|)x \cdot \nabla U\right) \\ &= \frac{d}{dr}\left(r^{n-1}\frac{dy}{dr}\right) + r^{n-1}f(x,U) + r^n g(r)\frac{dy}{dr} \\ &= \frac{n-2}{\beta(t)\beta'(t)}[\beta(t)]^{n-1}\left[u_0''(t) + \frac{1}{n-2}\beta(t)\beta'(t)f(x,U)\right. \\ &\left. + \beta(t)\beta'(t)g(\beta(t))\left(u_0'(t) - \frac{u_0(t)}{t}\right)\right], \end{aligned}$$

for any $t \geq t_1$.

We have obtained that

$$\begin{aligned} &|x|^{n-1}\left(\Delta U + f(x,U) + g(|x|)x \cdot \nabla U\right) \\ &\leq \frac{n-2}{\beta(t)\beta'(t)}[\beta(t)]^{n-1}\left[u_0''(t) + a(t)u_0(t)\right] = 0, \end{aligned}$$

where the functional coefficient a is given by (7).

Now, U is a positive super-solution of (3). Also, the trivial solution of (3) is its (nonnegative) sub-solution. According to Lemma 1, there exists a nonnegative solution u to (3), defined in $\overline{G}_B$. Since

$$(\Delta + g(|x|)x \cdot \nabla)(-u) = f(x,u) \geq 0,$$

the strong maximum principle ([15]) can be applied to $-u$. This means that the function $-u$ cannot attain a nonnegative maximum in a point of G_B unless it is constant. Since $-u$ is negative on $\{x : |x| = B\}$ and $-u(x) \leq 0$ throughout $\overline{G}_B$ as u is confined between 0 and a positive super-solution U, it follows that $-u$ cannot have zeros. We conclude that u is a positive solution of (3) that decays to 0 when $|x| \to +\infty$.

The proof is complete. ∎

Remark 3. A certain computational error has occurred in [1, p. 1209]. That is, the correct restriction on the function m from (4) is given by

$$\begin{aligned} & r^{n-2} \int_r^{+\infty} \tau^{3-n} \left| \frac{m(\tau)}{n-2} - [1 - q((n-2)\tau^{n-2})]g(\tau) \right| d\tau \\ & \leq q((n-2)r^{n-2}) - (n-2)r^{n-2} \int_{(n-2)r^{n-2}}^{+\infty} \frac{q^2(\tau)}{\tau^2} d\tau, \qquad r \geq A. \end{aligned}$$

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