

Spacelike Graphs with Parallel Mean Curvature *

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Abstract

We consider spacelike graphs Γ_f of simple products $(M \times N, g \times -h)$ where (M, g) and (N, h) are Riemannian manifolds and $f : M \rightarrow N$ is a smooth map. Under the condition of the Cheeger constant of M to be zero and some condition on the second fundamental form at infinity, we conclude that if $\Gamma_f \subset M \times N$ has parallel mean curvature H then $H = 0$. This holds trivially if M is closed. If M is the m -hyperbolic space then for any constant c , we describe an explicit foliation of $\mathbb{H}^m \times \mathbb{R}$ by hypersurfaces with constant mean curvature c .

1 Introduction

The problem of estimating the mean curvature of a surface of \mathbb{R}^3 described by a graph of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ was first introduced in 1955 by E. Heinz [12]. He proved that if f is defined on the disc $x^2 + y^2 < R^2$ and the mean curvature satisfies $\|H\| \geq c > 0$, where c is a constant, then $R \leq \frac{1}{c}$. So, if f is defined in all \mathbb{R}^2 and $\|H\|$ is constant, then $H = 0$. Ten years later this problem was extended and solved for the case of a map $f : \mathbb{R}^m \rightarrow \mathbb{R}$ by Chern [5] and independently, by Flanders [9]. In 1986, Jim Eells suggested to the author a generalization of this problem in her Ph.D thesis ([15], [16]). We recall the formulation of the problem.

Let (M, g) and (N, h) be two Riemannian manifolds of dimension m and n respectively, and $f : M \rightarrow N$ a smooth map. The graph of f , $\Gamma_f := \{(p, f(p)) : p \in M\}$ is a submanifold of $M \times N$ of dimension m . We take on $M \times N$ the product metric

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$g \times h$, and on Γ_f , the induced one \tilde{g} . Let H denote the mean curvature vector of Γ_f . On M it is defined the Cheeger constant

$$\mathfrak{h}(M) = \inf_D \frac{A(\partial D)}{V(D)}$$

where D ranges over all open submanifolds of M with compact closure in M and smooth boundary (see e.g. [4]), and $A(\partial D)$ and $V(D)$ are respectively the area of ∂D and the volume of D , with respect to the metric g . This constant is zero, if, for example, M is a closed manifold, or if (M, g) is a simple Riemannian manifold, i.e., there exists a diffeomorphism $\phi : (M, g) \rightarrow (\mathbb{R}^m, \langle, \rangle)$ onto \mathbb{R}^m such that $\lambda g \leq \phi^* \langle, \rangle \leq \mu g$ for some positive constants λ, μ . Then we have got

Theorem 1.1. ([15],[16]) *If Γ_f has parallel mean curvature with $c = \|H\|$, then for each oriented compact domain $D \subset M$ we have the isoperimetric inequality*

$$c \leq \frac{1}{m} \frac{A(\partial D)}{V(D)}.$$

Thus, $c \leq \frac{1}{m} \mathfrak{h}(M)$. In particular if (M, g) has zero Cheeger constant then Γ_f is in fact a minimal submanifold of $M \times N$.

In case N is oriented one dimensional with unit vector field "1", we do not need parallel mean curvature to obtain a formula

$$m \langle H, \nu \rangle_{(g \times h)} = \operatorname{div}_g \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}} \right) \quad (1.1)$$

where $\nu = \frac{(-\nabla f, 1)}{\sqrt{1 + \|\nabla f\|^2}}$ is a unit normal to Γ_f . This led to a more general result:

Theorem 1.2. ([15],[16]) *If N is oriented of dimension one and $f : M \rightarrow N$ is any map, then*

$$\min_D \|H\| \leq \frac{1}{m} \frac{A(\partial D)}{V(D)}. \quad (1.2)$$

This generalizes the inequality of Heinz-Chern-Flanders to graphs of functions $f : M \rightarrow \mathbb{R}$. We note that it is not possible to relax the assumption of H to be constant to $0 \leq H \leq C$, where C is a constant, without further assumptions, to conclude minimality. In fact we have the following example: Set $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = e^x$. Then $0 = \lim_{x \rightarrow \pm\infty} H < \frac{1}{2} \operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}} \right) = \frac{1}{2} e^x (1 + e^{2x})^{-\frac{3}{2}} \leq C$.

A more difficult kind of problem is the so-called Bernstein-type problems, that amounts to determine geometric conditions to conclude that a minimal submanifold must be totally geodesic. Recently Rosenberg [14] obtained a Bernstein type result for entire minimal graphs in $M^2 \times \mathbb{R}$. Alías, Dajczer and Ripoll have obtained in [2] a Bernstein-type result for surfaces in an ambient space a three-dimensional Riemannian manifold endowed with a homothetic Killing field, that includes the case of [14]:

Theorem 1.3. ([2]) *Let M^2 be a complete surface with Gauss curvature $K_M \geq 0$.*

- (i) *Any entire constant mean curvature graph in $M^2 \times \mathbb{R}$ is totally geodesic.*
- (ii) *If, in addition, $K_M(q) > 0$ at some point $q \in M$, then the graph is a slice.*

The proof is inspired in the ideas of Chern [6] of a proof of a Bernstein theorem in case $m = 2$, and consists on computing the Laplacian of the support function $\Theta = \langle \nu, e \rangle$, where ν is a globally defined unit normal vector field of the normal bundle, and e is a constant unit vector field tangent to the factor \mathbb{R} . The assumption of $K_M \geq 0$ is necessary, since in [7] and [13] it is shown the existence of non-trivial entire minimal graphs when $M^2 = \mathbb{H}^2$ is the hyperbolic plane.

In the case $M = \mathbb{H}^m$ the Cheeger constant is $(m - 1)$. We have an example constructed by the author in [15] of a graph in $\mathbb{H}^m \times \mathbb{R}$ with non-zero constant mean curvature:

Proposition 1.1. *Consider the hyperbolic space $\mathbb{H}^m = (B^m, g)$ where B^m is the unit open disk in \mathbb{R}^m with centre 0 and g is the complete metric given by $g = 4|dx|^2/(1 - |x|^2)^2$, of constant sectional curvature equal to -1 . Let $c \in [1 - m, m - 1]$ and $f_c : \mathbb{H}^m \rightarrow \mathbb{R}$ defined by:*

$$f_c(x) = \int_0^{r(x)} \frac{\frac{c}{(\sinh r)^{m-1}} \int_0^r (\sinh t)^{m-1} dt}{\sqrt{1 - \left(\frac{c}{(\sinh r)^{m-1}} \int_0^r (\sinh t)^{m-1} dt\right)^2}} dr,$$

where $r(x) = \log\left(\frac{1+|x|}{1-|x|}\right)$ is the distance function in \mathbb{H}^m to 0. Then f_c is smooth on all \mathbb{H}^m , and for each $d \in \mathbb{R}$, $\Gamma_{f_c+d} \subset \mathbb{H}^m \times \mathbb{R}$ has constant mean curvature given by $\|H\| = \frac{|c|}{m}$. In the particular case $m = 2$ and $c = 1$, f_c can be written as

$$f_c(x) = \int_0^{r(x)} \sqrt{\frac{1}{2}(\cosh r - 1)} dr = \frac{2}{\sqrt{1 - |x|^2}} - 2$$

In [16] we only give a brief explanation that this example exists in [15]. So we will give in the Appendix (section 3) the proof of Proposition 1.1, that reproduces the proof in [15]. Moreover, a slightly modified proof of this one gives a proof of Proposition 1.3 for the Lorentzian case. We also note the following:

Proposition 1.2. *For each $c \in [1 - m, m - 1]$, $\{\Gamma_{f_c+d}(x) : x \in \mathbb{H}^m, d \in \mathbb{R}\}$ defines a foliation of $\mathbb{H}^m \times \mathbb{R}$ by hypersurfaces with constant mean curvature c .*

Remark. If we fix d and let c to vary, then we also have a foliation of $\mathbb{H}^m \times \mathbb{R}$ by hypersurfaces with constant mean curvature c , with c varying on each leaf. This also holds for the Lorentzian case of Proposition 1.3. The author would like to thank the referee for pointing out this interesting detail.

In this note we study the case of $M \times N$ is endowed with the pseudo-Riemannian metric $g \times -h$. We abusively still call "minimal" submanifolds, the ones that satisfy $H = 0$. Note that a graph Γ_f is spacelike iff $f^*h \leq bg$ with $b : M \rightarrow \mathbb{R}$ a continuous locally Lipschitz function satisfying $0 \leq b(p) < 1, \forall p \in M$. If N is one-dimensional then $b = \|\nabla f\|$. In section 2 we will prove the following:

Theorem 1.4. *Let Γ_f be a spacelike graph with parallel mean curvature with $\|H\| = \sqrt{|\langle H, H \rangle|} = |c|$. Then $\|\nabla df\| \geq \sqrt{m}|c|(1-b)^2$, with equality iff $\nabla df = c = 0$. Furthermore, if $\mathfrak{h}(M) = 0$ and if $\|\nabla df\| = O((1-b)^2)$, then Γ_f is minimal. This is the case of M compact.*

Assume N is oriented one-dimensional and $f : M \rightarrow N$ defines a spacelike graph. Then $\nu = -\frac{(\nabla f, 1)}{\sqrt{1-\|\nabla f\|^2}}$ is a unit timelike vector field that spans the normal bundle, and defines a timelike direction. H is future directed if $H = -\|H\|\nu$, with $\|H\| \geq 0$.

Theorem 1.5. *Assume N is oriented one-dimensional and $f : M \rightarrow N$ defines a spacelike graph with future directed mean curvature. On a compact domain D , let $b_D = \max_D \|\nabla f\| = \max_D b$. Then*

$$\min_D \|H\| \leq \frac{1}{m} \frac{b_D}{\sqrt{1-b_D^2}} \frac{A(\partial D)}{V(D)}. \quad (1.3)$$

In particular, if (1) or (2) below holds:

- (1) Γ_f has constant mean curvature, $\mathfrak{h}(M) = 0$, and $b \leq C < 1$ for some constant C ;
- (2) $|H|$ and $b/(\sqrt{1-b^2})$ are both integrable on M ;

then Γ_f is a minimal spacelike hypersurface. This is the case of M compact.

If $\mathfrak{h}(M) \neq 0$, we have an example, very similar to the one of Proposition 1.1, except on a sign in some term of the denominator.

Proposition 1.3. *Let c be any constant and $f_c : \mathbb{H}^m \rightarrow \mathbb{R}$ defined by:*

$$f_c(x) = \int_0^{r(x)} \frac{\frac{c}{(\sinh r)^{m-1}} \int_0^r (\sinh t)^{m-1} dt}{\sqrt{1 + \left(\frac{c}{(\sinh r)^{m-1}} \int_0^r (\sinh t)^{m-1} dt \right)^2}} dr,$$

where $r(x) = \log\left(\frac{1+|x|}{1-|x|}\right)$ is the distance function in \mathbb{H}^m to 0. Then f_c is smooth on all \mathbb{H}^m , and for each $d \in \mathbb{R}$, $\Gamma_{f_c+d} \subset \mathbb{H}^m \times \mathbb{R}$ is a spacelike graph with constant mean curvature given by $\langle H, \nu \rangle = \frac{c}{m}$, and $\|\nabla f_c\|^2 \leq (c^2/(c^2+(m-1)^2)) < 1$. Furthermore, $\{\Gamma_{f_c+d}(x) : x \in \mathbb{H}^m, d \in \mathbb{R}\}$ defines a foliation of $\mathbb{H}^m \times \mathbb{R}$ by hypersurfaces with constant mean curvature c .

Examples of spacelike constant mean curvature $H = c$ hypersurfaces of \mathbb{R}_1^{n+1} are the hyperboloids, i.e. the graph of $f(x) = \sqrt{\frac{k^2}{m^2} \frac{1}{c^2} + \sum_{i=1}^k x_i^2}$, for $k = 1, \dots, n$. If $k = n$ this example and the ones of Propositions 1.1 and 1.3 are described as constant mean curvature graphs of a function $f : (M, g) \rightarrow \mathbb{R}$ of the form $f(x) = \phi(r(x))$, where $r(x)$ is the distance function in (M, g) to a fixed point and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function. Such f are in fact smooth maps because r^2 is so, and ϕ (unique for a chosen constant c) can be expressed in terms of r^2 .

It has been a relevant problem in General Relativity the study of the existence and uniqueness of space-like hypersurfaces with constant mean curvature in globally hyperbolic connected Lorentzian manifolds having a compact Cauchy surface

(GHLCS), and the existence of foliations by such hypersurfaces. Here we are treating only the case $M \times N$ with a simple product $g \times -h$. The metric of a GHLCS is conformally equivalent to a warped product metric. For example, if (M, g) is closed and \mathbb{R} is endowed with the metric dt^2 and $\alpha : M \rightarrow \mathbb{R}$ is any positive smooth function (the lapse function), then spacelike graphs Γ_f in $(M \times \mathbb{R}, g - \alpha^2 dt^2)$ exist with prescribed mean curvature $H : M \rightarrow \mathbb{R}$ for any function H satisfying $\int_M H \alpha \text{Vol}_M = 0$. These graphs are unique up to a constant (i.e, if Γ_f is a solution then Γ_{f+d} is also a solution). This was proved by Akutagawa ([1]) using the invertibility of the Laplace operator for closed M . In particular, if H is constant, then the submanifold must be minimal. On the other hand on Robertson-Walker spacetimes the slice hypersurfaces have constant mean curvature, and recently, Alías and Montiel [3] proved that under certain conditions on the warping function, these are the only closed examples. Gerhard [11] proved that GHLCS spaces can be foliated by constant mean curvature hypersurfaces if the big bang and the big crunch hypothesis is satisfied and if a time-like convergence condition holds.

2 Spacelike graphs

Now we take in the product $M \times N$ the pseudo-Riemannian metric $g \times -h$. If $f : M \rightarrow N$ then we denote by

$$\begin{aligned} \Gamma_f : M &\rightarrow (M \times N, g \times h) \\ p &\rightarrow (p, f(p)) \end{aligned}$$

and identify the set Γ_f with the embedding Γ_f , and let $\tilde{g} = \Gamma_f^*(g \times -h) = g - f^*h$. Assume that f satisfies $h(df(X), df(X)) < g(X, X)$. Then \tilde{g} is a Riemannian metric of Γ_f , that is Γ_f is a space-like submanifold of $(M \times N, g \times -h)$. Let H denote the mean curvature of Γ_f . Note that H is a time-like vector.

Let X_i a local o.n. frame of (M, g) and $\tilde{g}_{ij} = g(X_i, X_j) - h(df(X_i), df(X_j))$. Set

$$W = \text{trace}_{g-f^*h}(\nabla df) \in C^\infty(f^{-1}TN), \quad (2.1)$$

$$Z = \sum_{ij} \tilde{g}^{ij} h(W, df(X_i)) X_j \in C^\infty(TM) \quad (2.2)$$

The following formulas hold:

Lemma 2.1. *If Γ_f has parallel mean curvature, then:*

- (1) $mH = (Z, W + df(Z)) = (0, W)^\perp$.
- (2) $m^2 c^2 = \text{div}_g(Z)$, where $c^2 = -\langle H, H \rangle_{(g \times -h)}$.

Proof. The proof is very similar to the one of lemmas 1,2 and 3 of [16] with some adjustments on the sign of h . So we omit it. ■

Taking X_i a o.n. basis of $T_p M$ that diagonalizes f^*h , i.e, $df(X_i) = \lambda_i e_i$, for $i \leq k$ where e_i is an o.n. system of $T_{f(p)} N$, and $df(X_i) = 0$ for $i \geq k + 1$, we conclude that $ag \leq f^*h \leq bg$, and $\frac{1}{1-a}g \leq \tilde{g}^{-1} \leq \frac{1}{1-b}g$ where $a = \inf_i \lambda_i^2$ is the smallest eigenvalue of f^*h and $b = \sup_i \lambda_i^2$ the largest. If N is one-dimensional and $m \geq 2$, then $a = 0$ and $b = \|\nabla f\|$. If we reorder the eigenvalues $b = \lambda_1^2 \geq \lambda_2^2 \geq \dots \geq$

$\lambda_n^2 = a$, including repeated eigenvalues according to their multiplicity, by the Weyl's perturbation theorem each λ_i^2 is a continuous locally Lipschitz function. In particular $b : M \rightarrow [0, 1)$ is a continuous locally Lipschitz function.

From (2.1)-(2.2) we conclude:

Lemma 2.2. $\|Z\| \leq \frac{\sqrt{b}}{1-b}\|W\|$, and $\|W\| \leq \frac{\sqrt{m}}{1-b}\|\nabla df\|$.

Proof of Theorem 1.4.

Let Γ_f be a spacelike graph with parallel mean curvature. Using Lemma 2.1

$$\begin{aligned} -m^2c^2 &= \langle (Z, W + df(Z)), (Z, W + df(Z)) \rangle_{g \times -h} \\ &= \|Z\|^2 - \|W\|^2 - 2h(W, df(Z)) - \|df(Z)\|^2 \end{aligned}$$

Thus,

$$\begin{aligned} \|Z\|^2 &\leq -m^2c^2 + \|W\|^2 + 2\|W\|\|df(Z)\| + \|df(Z)\|^2 \\ &\leq -m^2c^2 + (\|W\| + \sqrt{b}\|Z\|)^2 \\ &\leq -m^2c^2 + \frac{m}{(1-b)^4}\|\nabla df\|^2 \end{aligned} \quad (2.3)$$

what implies the first assertion. If $\|\nabla df\|^2 = mc^2(1-b)^4$, then from (2.3) $Z = 0$. Consequently, by lemma 2.1(b), $c = 0$, and so $\nabla df = 0$. Now denote by \bar{n} the unit outward of ∂D . By lemma 2.1(2), Stokes and Lemma 2.2

$$m^2c^2V(D) = \int_{\partial D} g(Z, \bar{n}) \leq \int_{\partial D} \|Z\| \leq A(\partial D) \sup_{\bar{D}} \frac{\sqrt{mb}}{(1-b)^2} \|\nabla df\| \quad (2.4)$$

If $\|\nabla df\| = O((1-b)^2)$, there exist a constant $C > 0$ s.t. $\|\nabla df\| \leq C(1-b)^2$. Then, from (2.4), $m^2c^2 \leq C' \frac{A(\partial D)}{V(D)}$ for some constant C' and Theorem 1.4 is proved. ■

Now let us now assume N is oriented of dimension one with global vector field "1". If $p_i = h(df(X_i), 1)$, then $\tilde{g}_{ij} = \delta_{ij} - p_i p_j$, $\tilde{g}^{ij} = \delta_{ij} + \frac{p_i p_j}{(1-\|\nabla f\|^2)}$. Similarly to the Riemannian case [15], we can obtain a formula:

Lemma 2.3.

$$m\langle H, \nu \rangle = \operatorname{div}_g \left(\frac{\nabla f}{\sqrt{1 - \|\nabla f\|^2}} \right) \quad (2.5)$$

Proof of Theorem 1.5.

We obtain (1.3) by integration over D of (2.5) and use Stokes. (1) is an immediate consequence of the definition of $\mathfrak{h}(M)$, and (2) is a consequence of the extended theorem of Stokes due to Gaffney [10] applied to (2.5). ■

3 Appendix

3.1 Proof of Propositions 1.1 and 1.2

First we note that if f satisfies $c = (1.1)$ then it does so $f + d$, where d is a constant. The function $r(x) = \log\left(\frac{1+|x|}{1-|x|}\right) = 2 \tanh^{-1}(|x|)$ has the following properties: $\forall x \neq 0$, $\nabla r = \frac{1-|x|^2}{2|x|} \frac{x}{|x|}$, where the gradient of r is w.r.t. the metric g . Hence, $\|\nabla r\|_g = 1$ and $\Delta r = (m-1) \coth r$. We observe that r^2 is smooth. Let us write $f = \phi(r)$ with $\phi: \mathbb{R}_0^+ \rightarrow \mathbb{R}$. Then $\nabla f = \phi'(r) \nabla r$, and so (1.1) applied to f becomes equivalent to

$$\begin{aligned} c &= \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|_g^2}}\right) = \operatorname{div}\left(\frac{\phi'(r) \nabla r}{\sqrt{1 + (\phi'(r))^2}}\right) \\ &= \frac{\phi'(r) \Delta r}{\sqrt{1 + (\phi'(r))^2}} - \frac{(\phi'(r))^2 \phi''(r) \|\nabla r\|^2}{(1 + (\phi'(r))^2)^{\frac{3}{2}}} + \frac{\phi''(r) \|\nabla r\|_g^2}{\sqrt{1 + (\phi'(r))^2}} \end{aligned}$$

Using the above properties of r we get

$$\begin{aligned} c(1 + (\phi'(r))^2)^{\frac{3}{2}} &= \\ &= (m-1) \coth r (\phi'(r)) (1 + (\phi'(r))^2) - (\phi'(r))^2 \phi''(r) + \phi''(r) (1 + (\phi'(r))^2) \\ &= (m-1) \coth r (\phi'(r)) (1 + (\phi'(r))^2) + \phi''(r) \end{aligned}$$

With the substitution $w(r) = \phi'(r)$ the equation becomes

$$w' = c(1 + w^2)^{\frac{3}{2}} - (m-1) \coth r w (1 + w^2), \quad \forall r > 0 \quad (3.1)$$

The next step is to reduce this differential equation to a linear one through several changes of variables. First write (3.1) as

$$\frac{w w'}{(1 + w^2)^{\frac{3}{2}}} = c w - (m-1) \coth r \frac{w^2}{(1 + w^2)^{\frac{1}{2}}}$$

Let $y = \frac{1}{(1+w^2)^{\frac{1}{2}}} \in (0, 1]$. Then $w = \pm \frac{\sqrt{1-y^2}}{y}$. Assume first the sign $+$. Then

$$(3.1) \iff -y' = c \frac{\sqrt{1-y^2}}{y} - (m-1) \coth r \frac{1-y^2}{y^2} y.$$

Thus, $-yy' = c\sqrt{1-y^2} - (m-1) \coth r (1-y^2)$. Let $v = y^2 \in (0, 1]$. Then

$$(3.1) \iff -\frac{1}{2} \frac{v'}{\sqrt{1-v}} = c - (m-1) \coth r \sqrt{1-v}.$$

Finally, let $u = \sqrt{1-v} \in [0, 1)$. Hence

$$(3.1) \iff u' = c - (m-1) \coth r u \quad (3.2)$$

which equation is linear. Let us first suppose $c = 1$. Then, the general solution of (3.2) is given by

$$\begin{aligned} u(r) &= e^{-\int_{r_0}^r (m-1) \coth t dt} \left(\int_{r_0}^r e^{(m-1) \int_{r_0}^s (m-1) \coth t dt} ds + u_0 \right) \\ &= e^{-(m-1)(\log \sinh r - \log \sinh r_0)} \left(\int_{r_0}^r e^{(m-1)(\log \sinh s - \log \sinh r_0)} ds + u_0 \right) \\ &= \frac{(\sinh r_0)^{m-1}}{(\sinh r)^{m-1}} \left(\frac{1}{(\sinh r_0)^{m-1}} \int_{r_0}^r (\sinh s)^{m-1} ds + u_0 \right) \\ &= \frac{1}{(\sinh r)^{m-1}} \int_{r_0}^r (\sinh s)^{m-1} ds + u_0 \frac{(\sinh r_0)^{m-1}}{(\sinh r)^{m-1}}. \end{aligned}$$

Let us now put $r_0 = u_0 = 0$. Then we have

$$u(r) = \frac{1}{(\sinh r)^{m-1}} \int_0^r (\sinh s)^{m-1} ds, \quad \forall r > 0 \quad (3.3)$$

Next we prove that $u \in [0, 1)$ with $u(0) = 0$, and, moreover, that $\sup_{r \in (0, +\infty)} u(r) = \lim_{r \rightarrow +\infty} u(r) = \frac{1}{m-1}$. Obviously u is positive and with l'Hospital rule,

$$u(0) = \lim_{r \rightarrow 0} u(r) = \lim_{r \rightarrow 0} \frac{(\sinh r)^{m-1}}{(m-1)(\sinh r)^{m-2} \cosh r} = \lim_{r \rightarrow 0} \frac{\tanh r}{(m-1)} = 0.$$

If $u(r)$ attains a local maximum at some $r_0 \in (0, +\infty)$, then $u'(r_0) = 0$. From (3.2) we have $u(r_0) = \frac{\tanh r_0}{m-1}$. Thus, $u(r_0) < \frac{1}{m-1} \leq 1$. On the other hand, if there are no local maxima, then, necessarily, $\sup_{r \in (0, +\infty)} u(r) = \lim_{r \rightarrow +\infty} u(r)$. So only we have to calculate this limit. With partial integration

$$\begin{aligned} \int_0^r (\sinh s)^{m-1} ds &= \\ &= [\cosh s (\sinh s)^{m-2}]_0^r - (m-2) \int_0^r \cosh^2 s (\sinh s)^{m-3} ds \\ &= \cosh r (\sinh r)^{m-2} - (m-2) \int_0^r (1 + \sinh^2 s) (\sinh s)^{m-3} ds \\ &= \cosh r (\sinh r)^{m-2} - (m-2) \int_0^r (\sinh s)^{m-3} ds - (m-2) \int_0^r (\sinh s)^{m-1} ds. \end{aligned}$$

Thus $\int_0^r (\sinh s)^{m-1} ds = \frac{1}{m-1} \cosh r (\sinh r)^{m-2} - \frac{m-2}{m-1} \int_0^r (\sinh s)^{m-2} ds$, and

$$\frac{\int_0^r (\sinh s)^{m-1} ds}{(\sinh r)^{m-1}} = \frac{1}{m-1} \coth r - \frac{(m-2)}{(m-1) \sinh^2 r} \frac{\int_0^r (\sinh s)^{m-3} ds}{(\sinh r)^{m-3}}.$$

Since $\forall p$, $\frac{\int_0^r (\sinh s)^p ds}{(\sinh s)^p}$ is a bounded function on $r \in [0, +\infty)$, we have

$$\lim_{r \rightarrow +\infty} \frac{\int_0^r (\sinh s)^{m-1} ds}{(\sinh s)^{m-1}} = \frac{1}{m-1} \lim_{r \rightarrow +\infty} \coth r = \frac{1}{m-1}.$$

Therefore,

$$\sup_{r \in [0, +\infty)} u(r) = \frac{1}{m-1} \quad (3.4)$$

which is not a maximum. So, $0 \leq u(r) < \frac{1}{m-1}$, $\forall r \in [0, +\infty)$ and $u(r)$ satisfies (3.2) for $c = 1$. Let now c be any arbitrary constant. Then, the function $\tilde{u}(r) = cu(r)$ is a solution of (3.2), but we have to impose $\tilde{u}(r) \in [0, 1)$. From (3.4) we conclude that c must satisfy $0 \leq c \leq m - 1$. That is, $\forall 0 \leq c \leq m - 1$, the function

$$\tilde{u}(r) = c \frac{\int_0^r (\sinh s)^{m-1} ds}{(\sinh s)^{m-1}}$$

fulfills the condition specified in (3.2). In terms of the original function f , we have f given by the expression in the Prop.1.2. If we had chosen the sign $-$ for the expression of w we would get in (3.2) a $-c$ instead c and we would obtain \tilde{u} with a change of sign, or equivalently, the same expression as in the Proposition, with $c \in [-m + 1, 0]$. Obviously, f is smooth on $\mathbb{H}^m \sim \{0\}$. Let us now investigate the behaviour of f close to the origin. Near $t = 0$ we have the following Taylor expansions:

$$\begin{aligned} \sinh t &= t + \frac{t^3}{6} + O(t^5) = t(1 + \frac{t^2}{6} + O(t^4)) \\ (1+t)^m &= 1 + mt + \theta(t^2) \\ \frac{1}{\sqrt{1+t}} &= 1 - \frac{t}{2} + \theta(t^2), \quad \frac{1}{1-t} = 1 + t + \theta(t^2) \end{aligned}$$

where $\theta(t)$ and $O(t^k)$ are analytic functions of the form

$$\theta(t^k) = \sum_{n \geq 0} \frac{a^{k+n}}{(k+n)!} t^{k+n} \quad O(t^k) = \sum_{n \geq 0} \frac{a^{k+2n}}{(k+2n)!} t^{k+2n}$$

Then we have $\frac{1}{\sqrt{1+t^2}} = 1 - \frac{t^2}{2} + \theta(t^4)$, $\frac{1}{1-t^2} = 1 + t^2 + \theta(t^4)$, and

$$(\sinh t)^{m-1} = t^{m-1} (1 + \frac{t^2}{6} + O(t^4))^{m-1} = t^{m-1} (1 + \frac{(m-1)}{6} t^2) + O(t^{m+3}).$$

Hence

$$\begin{aligned} \frac{1}{(\sinh s)^{m-1}} \int_0^s (\sinh t)^{m-1} dt &= \frac{\frac{s^m}{m} + \frac{m-1}{m-2} \frac{s^{m+2}}{6} + O(s^{m+4})}{s^{m-1} (1 + \frac{(m-1)}{6} s^2 + O(s^4))} = \\ &= \frac{\frac{s}{m} + \frac{m-1}{m-2} \frac{s^3}{6} + O(s^5)}{(1 + \frac{(m-1)}{6} s^2 + O(s^4))} \\ &= \left(\frac{s}{m} + \frac{m-1}{m-2} \frac{s^3}{6} + O(s^5) \right) \left(1 - s^2 \left(\frac{m-1}{6} + O(s^2) \right) + O(s^4) \right) \\ &= s \left(1 - \frac{(m-1)}{6} s^2 \right) \left(\frac{1}{m} + \frac{(m-1)}{(m+2)} \frac{s^2}{6} \right) + O(s^5) \\ &= \frac{s}{m} \left(1 - \frac{(m-1)}{(m+2)} \frac{s^2}{3} \right) + O(s^5) \end{aligned}$$

For A close to zero, $\frac{A}{\sqrt{1-A^2}} = A(1 + \frac{1}{2}A^2) + O(A^5)$. Putting

$$A = \frac{c}{(\sinh s)^{m-1}} \int_0^s (\sinh t)^{m-1} dt = c \frac{s}{m} \left(1 - \frac{(m-1)}{(m+2)} \frac{s^2}{3} \right) + O(s^5)$$

we have $O(A^5) = O(s^5)$ and

$$\begin{aligned} \frac{A}{\sqrt{1-A^2}} &= \\ &= \left(s \frac{c}{m} \left(1 - \frac{(m-1)}{(m+2)} \frac{s^2}{3} \right) + O(s^5) \right) \left(1 + \frac{1}{2} \left(c \frac{s}{m} \left(1 - \frac{(m-1)}{(m+2)} \frac{s^2}{3} \right) + O(s^5) \right)^2 \right) + O(s^5) \\ &= s \frac{c}{m} \left(1 + s^2 \left(\frac{c^2}{2m^2} - \frac{(m-1)}{3(m+2)} \right) \right) + O(s^5). \end{aligned}$$

Therefore

$$\int_0^r \frac{A}{\sqrt{1-A^2}} ds = \frac{c}{m} \frac{r^2}{2} + \frac{c}{m} \frac{r^4}{4} \left(\frac{c^2}{2m^2} - \frac{(m-1)}{3(m+2)} \right) + O(r^6) = \frac{c}{m} \frac{r^2}{2} + O(r^4)$$

Consequently

$$f(x) = \int_0^{r(x)} \frac{A}{\sqrt{1-A^2}} ds = \frac{c}{m} \frac{r^2(x)}{2} + O(r^4(x)).$$

Since $r^2(x)$ is smooth on all \mathbb{H}^m , we conclude that $f(x)$ is, too.

Finally, for each $x \neq 0$ fixed, the function $c \rightarrow f_c(x)$ has non-zero derivative. Moreover, $\Gamma_{f_c+d}(x) = \Gamma_{f_{c+d'}}(x')$ implies $x = x'$ and $d = d'$. So we have two possible foliations, either varying c or d . Note that $O(r^4)$ also depends on c . ■

3.2 Proof of Proposition 1.3

We solve $c = (2.5)$ for $f = \phi(r)$. In this case we follow the previous proof, with the following replacements:

$$\begin{aligned} c &= \operatorname{div} \left(\frac{\phi'(r) \nabla r}{\sqrt{1 - (\phi'(r))^2}} \right) \\ w &= \phi'(r), \quad |w| < 1 & w' &= c(1 - w^2)^{\frac{3}{2}} - (m-1) \coth r w (1 - w^2) \\ y &= \frac{1}{\sqrt{(1 - w^2)}} \in [1, +\infty) & v &= y^2 \in [1, +\infty) \\ u &= \sqrt{v - 1} \in [0, +\infty) & u' &= c - (m-1) \coth r u \end{aligned}$$

Thus $u(r)$ is the same function as in the Riemannian case, but now we do not have any restriction on the range of values of $u(r)$. This implies we may choose first $u(r)$ as defined in (3.3), that corresponds to take $c = 1$, and next take $\tilde{u} = cu$ for any constant c with no restrictions on the chosen c . Finally, the proof that f is smooth close the origin we use $\frac{A}{\sqrt{1+A^2}} = A(1 - \frac{1}{2}A^2) + O(A^5)$ obtaining as well

$$f(x) = \int_0^{r(x)} \frac{A}{\sqrt{1+A^2}} ds = \frac{c}{m} \frac{r^2(x)}{2} + O(r^4(x)).$$

and proving its smoothness.

We also note that the hyperboloid with $k = n$ is obtained in the same way, by taking $r(x) = \|x\|$ the Euclidean norm. ■

References

- [1] K. Akutagawa, *A note on spacelike hypersurfaces with prescribed mean curvature in a spatially closed globally static Lorentzian manifold*, Mem. Fac. Sci. Kyushu Univ. Ser. A **40** (1986), no. 2, 119–123.
- [2] L.J. Alías; M. Dajczer; J. Ripoll, *A Bernstein-type theorem for Riemannian manifolds with a Killing field*, Ann. Global Anal. Geom. **31** (2007), no.4, 363–373.
- [3] L.J. Alías; S. Montiel, *Uniqueness of spacelike hypersurfaces with constant mean curvature in generalized Robertson-Walker spacetimes*. Differential geometry, Valencia, 2001, 59–69, World Sci. Publishing, River Edge, NJ, 2002.
- [4] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984.
- [5] S.S. Chern, *On the curvatures of a piece of hypersurface in euclidean space*. Abh. Math. Sem. Univ. Hamburg **29** (1965), 77–91.
- [6] S.S. Chern, *Simple proofs of two theorems on minimal surfaces*. Enseignement Math. (2) **15** (1969) 53–61.
- [7] D.M. Duc; N.V. Hieu, *Graphs with prescribed mean curvature on Poincaré disk*, Bull. London Math. Soc. **27** (1995), no. 4, 353–358.
- [8] D.M. Duc; I.M.C. Salavessa, *On a class of graphs with prescribed mean curvature*, Manuscripta Math. **82** (1994), no. 3-4, 227–239.
- [9] H. Flanders, *Remark on mean curvature*, J. London Math. Soc. **41** (1966), 364–366.
- [10] M.P. Gaffney, *A special Stokes's theorem for complete Riemannian manifolds*. Ann. of Math. (2) **60**, (1954). 140–145.
- [11] C. Gerhardt, *H-surfaces in Lorentzian manifolds*, Comm. Math. Phys. **89** (1983), no. 4, 523–553.
- [12] E. Heinz, *Über Flächen mit eineindeutiger Projektion auf eine Ebene, deren Krümmungen durch Ungleichungen eingeschränkt sind*. Math. Ann. **129** (1955), 451–454.
- [13] B. Nelli; H. Rosenberg, *Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$* , Bull. Braz. Math. Soc. (N.S.) **33** (2002), no. 2, 263–292.
- [14] H. Rosenberg, *Minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$* . Illinois J. Math. **46** (2002), no. 4, 1177–1195.

- [15] I.M.C. Salavessa, *Graphs with parallel mean curvature and a variational problem in conformal geometry*, Ph.D. Thesis, University of Warwick, 1987
- [16] I.M.C. Salavessa, *Graphs with parallel mean curvature*. Proc. Amer. Math. Soc. **107** (1989), no. 2, 449–458.

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