On the stability of a mixed n-dimensional quadratic functional equation

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Abstract

In this paper, we investigate the modified Hyers-Ulam stability of a mixed n-dimensional quadratic functional equation in Banach spaces and also Banach modules over a Banach algebra and a C^*-algebra. Finally, we study the stability using the alternative fixed point of the functional equation in Banach spaces:

$$_{n-2}C_{m-2} f(\sum_{j=1}^{n} x_j) +_{n-2} C_{m-1} \sum_{i=1}^{n} f(x_i) = \sum_{1 \le i_1 < \cdots < i_m \le n} f(x_{i_1} + \cdots + x_{i_m}),$$

for all x_j $(j = 1, \cdots, n)$ where $n \ge 3$ is an integer number and $2 \le m \le n-1$.

1 Introduction

In 1940, the problem of stability of functional equations was originated by Ulam [24] as follows: Under what condition does there exist an additive mapping near an approximately additive mapping ?

The first partial solution to Ulam's question was provided by D. H. Hyers [7]. Let X and Y are Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Hyers showed that if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \le \epsilon$$

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for a given fixed $\epsilon \geq 0$ and for all $x, y \in X$, then the limit

$$a(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$ and $a : X \to Y$ is the unique additive function such that

$$\| f(x) - a(x) \| \leq \epsilon$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

Hyers's theorem was generalized in various directions. In particular, thirty seven years after Hyers's Theorem, Th.M.Rassias provided a generalization of Hyers's result by allowing the Cauchy difference to be unbounded; see [15]. He proved the following theorem: if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| x \|^p)$$

for some $\theta \geq 0$, $0 \leq p < 1$, and for all $x, y \in X$, then there exists a unique additive function such that

$$\| f(x) - a(x) \| \leq \frac{2\theta}{2 - 2^p} \| x \|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

Th.M. Rassias result provided a generalization of Hyers Theorem, a fact which rekindled interest in the study of stability of functional equations. Taking this fact into consideration the Hyers-Ulam stability is called Hyers-Ulam-Rassias stability. In 1990, Th.M.Rassias during the 27th International Symposium on Functional Equations asked the question whether an extension of his Theorem can be proved for all positive real numbers p that are greater or equal to one. A year later in 1991, Gajda provided an affirmative solution to Rassias's question in the case the number p is greater than one; see [5].

During the last two decades several results for the Hyers-Ulam-Rassias stability of functional equations have been proved by several mathematicians worldwide in the study of several important functional equations of several variables. Găvruta [6] following Rassias's approach for the unbounded Cauchy difference provided a further generalization.

The quadratic function $f(x) = cx^2$ $(c \in \mathbb{R})$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Hence this question is called the quadratic functional equation, and every solution of the quadratic equation (1.1) is called a quadratic function.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was proved by Skof [23] for functions $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. In [3], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated; see [19], [20], and [21]. Recently, Bae and Park investigated that the generalized Hyers-Ulam-Rassias stability of $n-$dimensional quadratic functional equations in Banach modules over a C^*-algebra and unitary Banach algebra; see [1].

In particular, Trif [14] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n \cdot_{n-2} C_{m-2} f(\frac{x_1 + \cdots + x_n}{n}) +_{n-2} C_{m-1} \sum_{i=1}^{n} f(x_i) \tag{1.2}$$

$$= k \sum_{1 \le i_1 < \cdots < i_m \le n} f(\frac{x_{i_1} + \cdots + x_{i_m}}{k}),$$

for all $x_1, \cdots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x)+f(y)$ for all $x, y \in V$. He also proved the stability of the functional equation (1.2); see [14]. Note that the notation ${}_nC_k$ is defined by ${}_nC_k = \frac{n!}{(n-k)!\,k!}$.

In this paper, we consider the following functional equation:

$$_{n-2}C_{m-2} f(\sum_{j=1}^{n} x_j) +_{n-2} C_{m-1} \sum_{i=1}^{n} f(x_i) \tag{1.3}$$

$$= \sum_{1 \le i_1 < \cdots < i_m \le n} f(x_{i_1} + \cdots + x_{i_m}),$$

where $n \ge 3$ is an integer number and $2 \le m \le n-1$. Then for any $m \in \{2, \cdots, n-1\}$, we will show that the even mapping f satisfying the equation (1.3) is quadratic, investigate the generalized Hyers-Ulam-Rassias stability of the mixed n-dimensional functional equation in Banach spaces and also extend to Banach modules over a C^*-algebra and a unital Banach algebra. Finally, we study the stability using the alternative fixed point of the functional equation in Banach spaces.

2 A Mixed n-dimensional quadratic mapping

Lemma 2.1. *Let $n \ge 3$ be an integer, and let X, Y be vector spaces. For any $m \in \{2, \cdots, n-1\}$, suppose an even mapping $f : X \to Y$ is defined by*

$$_{n-2}C_{m-2} f(\sum_{j=1}^{n} x_j) +_{n-2} C_{m-1} \sum_{i=1}^{n} f(x_i) \tag{2.1}$$

$$= \sum_{1 \le i_1 < \cdots < i_m \le n} f(x_{i_1} + \cdots + x_{i_m}),$$

for all $x_1, \cdots, x_n \in X$. Then f is quadratic.

Proof. By letting $x_1 = \cdots = x_n = 0$ in (2.1), we have

$$({}_{n-2}C_{m-2} + n\,{}_{n-2}C_{m-1} -_n C_m) f(0) = 0$$

Then we obtain

$$\frac{(m-1)(n-1)!}{m!\,(n-m-1)!} f(0) = 0\,.$$

Since $n \geq 3$, $f(0) = 0$. Also, letting $x_1 = x$, $x_2 = -y$, $x_3 = y$, and $x_k = 0\,(4 \leq k \leq n)$ in (2.1), we get

$$_{n-2}C_{m-2}f(x) +_{n-2} C_{m-1}f(x) + 2\,_{n-2}C_{m-1}f(y)$$

$$=_{n-3} C_{m-2}(f(x+y) + f(x-y)) + (_{n-3}C_{m-3} +_{n-3} C_{m-1})f(x) + 2\,_{n-3}C_{m-1}f(y)\,.$$

Since $_nC_{r+1} = \,_{n-1}C_r + \,_{n-1}C_{r+1}$, then

$$\begin{aligned} &_{n-3}C_{m-2}(f(x+y) + f(x-y)) \\ &= (_{n-2}C_{m-2} +_{n-2} C_{m-1} -_{n-3} C_{m-3} -_{n-3} C_{m-1})f(x) \\ &\quad + 2(_{n-2}C_{m-1} -_{n-3} C_{m-1})f(y) \\ &= 2_{n-3}C_{m-2}(f(x) + f(y)). \end{aligned}$$

Hence we may have

$$_{n-3}C_{m-2}(f(x+y) + f(x-y)) = 2\,_{n-3}C_{m-2}(f(x) + f(y))\,,$$

that is, f is quadratic, as desired. ■

The mapping $f : X \to Y$ as in the Lemma 2.1 is called a $n-$*dimensional quadratic mapping.*

3 Stability of a mixed n-dimensional quadratic mapping with $2 \leq m \leq n-1$

Throughout in this section, let X be a normed vector space with norm $\|\cdot\|$ and Y be a Banach space with norm $\|\cdot\|$. Let $n \geq 3$ be an integer number and $2 \leq m \leq n-1$.

For the given mapping $f : X \to Y$, we define

$$D_m f(x_1, \cdots, x_n) := \,_{n-2}C_{m-2}f(\sum_{j=1}^{n} x_j) + \,_{n-2}C_{m-1}\sum_{i=1}^{n} f(x_i) \tag{3.1}$$

$$- \sum_{1 \leq i_1 < \cdots < i_m \leq n} f(x_{i_1} + \cdots + x_{i_m})\,,$$

for all $x_1, \cdots, x_n \in X$.

In this section, for any $m \in \{2, \cdots, n-1\}$, we will investigate the generalized Hyers-Ulam-Rassias stability of the equation (3.1). Before proceeding the proofs, we note that

$$_nC_r = 0\,,$$

when $n < r$, or $r < 0$. Also, we denote $_0C_0 = 1$.

Theorem 3.1. *Let $n \geq 3$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 2^{-2j}\phi(2^j x_1, \cdots, 2^j x_n) < \infty\,, \tag{3.2}$$

$$\| D_m f(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n), \tag{3.3}$$

for all $x_1, \cdots, x_n \in X$. *Then for any* $m \in \{2, \cdots, n-1\}$, *there exists a unique* n*-dimensional quadratic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x) \| \leq \frac{1}{4\,{}_{n-3}C_{m-2}} \widetilde{\phi}(x, -x, x, 0, \cdots, 0), \tag{3.4}$$

for all $x \in X$.

Proof. By letting $x_1 = x$, $x_2 = -x$, $x_3 = x$, and $x_k = 0\,(4 \leq k \leq n)$ in (3.3), we have

$$\begin{aligned} \| ({}_{n-2}C_{m-2} &+ 3 \cdot {}_{n-2} C_{m-1} - 3 \cdot {}_{n-3} C_{m-1} - {}_{n-3}C_{m-3}) f(x) \\ &- {}_{n-3}C_{m-2} f(2x) \| \leq \phi(x, -x, x, 0, \cdots, 0), \end{aligned}$$

for all $x \in X$. Since

$${}_{n-2}C_{m-2} + 3 \cdot {}_{n-2} C_{m-1} - 3 \cdot {}_{n-3} C_{m-1} - {}_{n-3}C_{m-3} = 4 \cdot {}_{n-3} C_{m-2},$$

we get

$$\| f(x) - 2^{-2} f(2x) \| \leq \frac{1}{4 \cdot {}_{n-3} C_{m-2}} \phi(x, -x, x, 0, \cdots, 0), \tag{3.5}$$

for all $x \in X$.

Inductively, if x is replaced by $2x$ and apply to transitive inequality, we may have

$$\begin{aligned} \| f(x) &- (\frac{1}{2})^{2s} f(2^s x) \| \\ &\leq \frac{1}{4 \cdot {}_{n-3} C_{m-2}} \sum_{k=0}^{s-1} (\frac{1}{2})^{2k} \phi(2^k x, -2^k x, 2^k x, 0, \cdots, 0), \end{aligned}$$

for all $x \in X$ and all positive integer s. Also, for all integers $r > l > 0$, we have

$$\begin{aligned} (*) \qquad & \| (\frac{1}{2})^{2r} f(2^r x) - (\frac{1}{2})^{2l} f(2^l x) \| \\ \leq \ & \frac{1}{4 \cdot {}_{n-3} C_{m-2}} \sum_{k=l}^{r-1} (\frac{1}{2})^{2k} \phi(2^k x, -2^k x, 2^k x, 0, \cdots, 0), \end{aligned}$$

for all $x \in X$.

Then the sequence $\{(\frac{1}{2})^{2s} f(2^s x)\}$ is a Cauchy sequence in a Banach space Y. Hence we may define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{s \to \infty} 2^{-2s} f(2^s x),$$

for all $x \in X$. By the definition of $D_m Q(x_1, \cdots, x_n)$,

$$\begin{aligned} \| D_m Q(x_1, \cdots, x_n) \| &= \lim_{s \to \infty} (\frac{1}{2})^{2s} \| D_m f(2^s x_1, \cdots, 2^s x_n) \| \\ &\leq \lim_{s \to \infty} (\frac{1}{2})^{2s} \phi(2^s x_1, \cdots, 2^s x_n) = 0, \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. That is, $D_m Q(x_1, \cdots, x_n) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. Also, letting $l = 0$ and passing the limit $r \to \infty$ in (*), we get the inequality (3.4).

Now, let $Q' : X \to Y$ be another n-dimensional quadratic mapping satisfying (3.4). Then we have

$$\begin{aligned} \| Q(x) - Q'(x) \| &= 2^{-2r} \| Q(2^r x) - Q'(2^r x) \| \\ &\leq \frac{2 \cdot 2^{-2r}}{4 \cdot {}_{n-3}C_{m-2}} \widetilde{\phi}(2^r x, -2^r x, 2^r x, 0, \cdots, 0), \end{aligned}$$

for all $x \in X$. As $r \to \infty$, we may conclude that $Q(x) = Q'(x)$, for all $x \in X$. Thus such an n-dimensional quadratic mapping $Q : X \to Y$ is unique. ∎

Theorem 3.2. *Let $n \geq 3$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 2^{2j} \phi(2^{-j} x_1, \cdots, 2^{-j} x_n) < \infty, \tag{3.6}$$

$$\| D_m f(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n), \tag{3.7}$$

for all $x_1, \cdots, x_n \in X$. Then for any $m \in \{2, \cdots, n-1\}$, there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{{}_{n-3}C_{m-2}} \widetilde{\phi}(\frac{1}{2}x, -\frac{1}{2}x, \frac{1}{2}x, 0, \cdots, 0), \tag{3.8}$$

for all $x \in X$.

Proof. If x is replaced by $\frac{1}{2}x$ in the inequality (3.5), then the proof follows from the proof of Theorem 3.1. ∎

Corollary 3.3. *Let $p \neq 2$ and θ be positive real numbers, let $n \geq 3$ be an integer and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and for each integer m such that $2 \leq m \leq n-1$,*

$$\| D_m f(x_1, \cdots, x_n) \| \leq \theta \sum_{i=1}^{n} ||x_i||^p,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{2 \cdot {}_{n-3}C_{m-2}} \cdot \frac{\theta}{|4 - 2^p|} ||x||^p,$$

for all $x \in X$.

Proof. Let

$$\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p,$$

and then apply to Theorem 3.1 when $0 < p < 2$, or apply to Theorem 3.2 when $p > 2$. ∎

4 Another stability of a mixed n-dimensional quadratic mapping with special cases $m = 2$ and $m = n - 1$

We will start with $m = 2$ in the equation (3.1). Then the equation $D_m f(x_1, \cdots, x_n)$ can be reduced to the following form

$$\begin{aligned} D_2 f(x_1, \cdots, x_n) &:= f(\sum_{j=1}^{n} x_j) + (n-2)\sum_{i=1}^{n} f(x_i) \\ &- \sum_{1\le i_1 \le i_2 \le n} f(x_{i_1} + x_{i_2}), \end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

Theorem 4.1. *Let $n \geq 3$ be an integer number and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 2^{-2j} \phi(2^j x_1, \cdots, 2^j x_n) < \infty, \tag{4.1}$$

$$\| D_2 f(x_1, \cdots, x_n) \| \le \phi(x_1, \cdots, x_n), \tag{4.2}$$

for all $x_1, \cdots, x_n \in X$. Then for any odd integer t with $3 \le t \le n$, there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \le \frac{1}{(t-1)^2} \widetilde{\phi}(\underbrace{x, -x, \cdots, -x, x}_{t-terms}, 0, \cdots, 0), \tag{4.3}$$

for all $x \in X$.

Proof. For any odd integer $3 \le t \le n$, let

$$x_j = \begin{cases} (-1)^{j-1} x & 1 \le j \le t, \\ 0 & t+1 \le j \le n . \end{cases}$$

in the inequality (4.2), we have

$$\begin{aligned} & \| D_2 f(\underbrace{x, -x, \cdots, -x, x}_{t-terms}, 0, \cdots, 0) \| \\ = & \| f(\sum_{j=1}^{n} x_j) + (n-2)\sum_{i=1}^{n} f(x_i) - \sum_{1\le i_1 < i_2 \le n} f(x_{i_1} + x_{i_2}) \| \\ = & \| f(x) + (n-2)\Big(\frac{t+1}{2} f(x) + \frac{t-1}{2} f(-x)\Big) \\ & -\Big({}_{\frac{t+1}{2}}C_2 f(2x) + {}_{\frac{t-1}{2}}C_2 f(-2x) \\ & + {}_{\frac{t+1}{2}}C_1 \cdot {}_{n-t}C_1 f(x) + {}_{\frac{t-1}{2}}C_1 \cdot {}_{n-t}C_1 f(-x)\Big) \| \end{aligned}$$

$$\begin{aligned}
&= \| \, (t(n-2)+1)f(x) \\
&\quad -\Big([\frac{\frac{t+1}{2}(\frac{t+1}{2}-1)}{2} + \frac{\frac{t-1}{2}(\frac{t-1}{2}-1)}{2}]f(2x) \\
&\quad +[\frac{t+1}{2}\cdot(n-t) + \frac{t-1}{2}\cdot(n-t)]f(x) \Big) \, \| \\
&= \| \, (t(n-2)-t(n-t)+1)f(x) - \frac{1}{4}(t-1)^2 f(2x) \, \| \\
&\le \phi(\underbrace{x,-x,\cdots,-x,x}_{t-terms},0,\cdots,0)\,,
\end{aligned}$$

that is,

$$\| \, f(x) - (\frac{1}{2})^2 f(2x) \, \| \le \frac{1}{(t-1)^2}\phi(\underbrace{x,-x,\cdots,-x,x}_{t-terms},0,\cdots,0)\,, \tag{4.4}$$

for all $x \in X$. Remains follow from the proof of Theorem 3.1. ∎

Theorem 4.2. *Let $n \ge 3$ be an integer number and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0,\infty)$ such that*

$$\widetilde{\phi}(x_1,\cdots,x_n) := \sum_{j=0}^{\infty} 2^{-2j}\phi(2^j x_1,\cdots,2^j x_n) < \infty\,, \tag{4.5}$$

$$\| \, D_2 f(x_1,\cdots,x_n) \, \| \le \phi(x_1,\cdots,x_n)\,, \tag{4.6}$$

for all $x_1,\cdots,x_n \in X$. Then for any even integer t with $4 \le t \le n$, there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| \, f(x) - Q(x) \, \| \le \frac{1}{t(t-2)}\widetilde{\phi}(\underbrace{x,-x,\cdots,x,-x}_{t-terms},0,\cdots,0)\,, \tag{4.7}$$

for all $x \in X$.

Proof. For any even integer t such that $4 \le t \le n$, let

$$x_j = \begin{cases} (-1)^{j-1}x & 1 \le j \le t, \\ 0 & t+1 \le j \le n\,. \end{cases}$$

in the inequality (4.6), we have

$$\| \, t(t-2)f(x) - \frac{1}{4}t(t-2)f(2x) \, \| \le \phi(\underbrace{x,-x,\cdots,x,-x}_{t-terms},0,\cdots,0)\,.$$

Then we have

$$\| \, f(x) - (\frac{1}{2})^2 f(2x) \, \| \le \frac{1}{t(t-2)}\phi(\underbrace{x,-x,\cdots,x,-x}_{t-terms},0,\cdots,0)\,, \tag{4.8}$$

for all $x \in X$. Similar to the proof of Theorem 3.1, we have the desired result when t is even. ∎

When $m = n-1$, the equation $D_m f(x_1, \cdots, x_n)$ in (3.1) forms

$$D_{n-1}f(x_1, \cdots, x_n) := \\ (n-2)f(\sum_{j=1}^{n} x_j) \; + \; \sum_{i=1}^{n} f(x_i) \; - \sum_{1\le i_1<\cdots<i_{n-1}\le n} f(x_{i_1} + \cdots + x_{i_{n-1}}),$$

for all $x_1, \cdots, x_n \in X$.

We will consider two cases where $n \geq 3$ is odd and $n \geq 4$ is any integer.

Theorem 4.3. *Let $n \geq 3$ be odd and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^{-j}\phi(2^j x_1, \cdots, 2^j x_n) < \infty, \tag{4.9}$$

$$\| D_{n-1}f(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n), \tag{4.10}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{2(n-1)}\widetilde{\phi}(x, -x, x, -x, \cdots, -x, x), \tag{4.11}$$

for all $x \in X$.

Proof. For each $k = 1, \cdots, n$, letting $x_k = (-1)^{k-1}x$ in (4.10), we have

$$\| 2(n-1)f(x) - \frac{n-1}{2}f(2x) \| \leq \phi(x, -x, x, -x, \cdots, -x, x),$$

for all $x \in X$. Then we write

$$\| f(x) - \frac{1}{4}f(2x) \| \leq \frac{1}{2(n-1)}\phi(x, -x, x, -x, \cdots, -x, x), \tag{4.12}$$

for all $x \in X$. The remains follow from the proof of Theorem 3.1. ∎

Now, we may assume $n \geq 4$ is an integer.

Theorem 4.4. *Let $n \geq 4$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^{-j}\phi(2^j x_1, \cdots, 2^j x_n) < \infty, \tag{4.13}$$

$$\| D_{n-1}f(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n), \tag{4.14}$$

for all $x_1, \cdots, x_n \in X$. Then for any integer m such that $4 \leq 2m \leq n$, there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{4(m-1)}\widetilde{\phi}(\underbrace{2x, -x, x, \cdots, x, -x}_{2m-terms}, 0, \cdots, 0), \tag{4.15}$$

for all $x \in X$.

Proof. By letting $x_1 = 2x$, $x_k = (-1)^{k+1}x$, $(k = 2, \cdots, 2m)$, and $x_k = 0$, $(2m+1 \leq k \leq n)$ in (4.14), we have

$$\begin{aligned} \| \, (n-2)f(x) + f(2x) \quad &+ \quad (2m-1)f(x) \\ &- \quad (mf(2x) + f(x) + (n-2m)f(x)) \, \| \\ \leq \phi(\underbrace{2x, -x, x, \cdots, x, -x}_{2m-terms}, 0, \quad &\cdots \quad , 0)\,, \end{aligned}$$

for all $x \in X$. Then we have

$$\| \, f(x) - \frac{1}{4}f(2x) \, \| \leq \frac{1}{4(m-1)}\phi(\underbrace{2x, -x, x, \cdots, x, -x}_{2m-terms}, 0, \cdots, 0)\,, \tag{4.16}$$

for all $x \in X$. Similar to the proof of Theorem 3.1, we have the desired results. ∎

Note that Theorem 4.4 remains valid if $n \geq 4$ is either odd or even.

Remark 4.5. Similar to section 3, that is, Theorem 3.2 can be obtained from Theorem 3.1 by replacing x by $\frac{1}{2}x$, in section 4 similar Theorems can be obtained.

5 Results in Banach modules over a Banach algebra

Throughout this section, let B be a unital Banach $*$-algebra with norm $| \; |$ and $B_1 = \{a \in B \mid |a| = 1\}$, let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach modules with norms $\| \; \|$ and $\| \; \|$, respectively, and let

$$\varphi : [{}_B\mathbb{B}_1 \setminus \{0\}]^n \to \mathbb{R}$$

be the function such that

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 2^{-2j}\varphi(2^j x_1, \cdots, 2^j x_n) < \infty\,, \tag{5.1}$$

for all $x_1, \cdots, x_n \in_B \mathbb{B}_1 \backslash \{0\}$.

Definition 5.1. An n-dimensional quadratic mapping

$$Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$$

is called *n-dimensional $B-$quadratic* if $Q(ax) = a^2Q(x)$ for all $a \in B$ and all $x \in_B \mathbb{B}_1$.

Definition 5.2. For $a \in B$, let $b = aa^*,\ a^*a,$ or $(aa^* + a^*a)/2$. An n-dimensional quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ is called *n-dimensional $B_{sa}-$quadratic* if $Q(ax) = bQ(x)$, for all $a \in B$, and all $x \in_B \mathbb{B}_1$.

Since Banach spaces ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ are considered as Banach modules over $B := \mathbb{C}$, the $B_{sa}-$quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ implies $Q(ax) = |a|^2Q(x)$, for all $a \in \mathbb{C}$.

We define the *approximate remainder* $(D_m)_a f$ for a mapping $f :_B \mathbb{B}_1 \rightarrow_B \mathbb{B}_2$,

$$(D_m)_a f(x_1, \cdots, x_n) := {}_{n-2}C_{m-2} f(\sum_{j=1}^{n} a x_j) + {}_{n-2}C_{m-1} \sum_{i=1}^{n} f(a x_i)$$

$$-b \sum_{1 \leq i_1 < \cdots < i_m \leq n} f(x_{i_1} + \cdots + x_{i_m}) ,$$

for all $x_1, \cdots, x_n \in_B \mathbb{B}_1$.

Theorem 5.3. *Let $f :_B \mathbb{B}_1 \rightarrow_B \mathbb{B}_2$ be a mapping with $f(0) = 0$ for the case (5.1) which there is a mapping $\varphi :_B \mathbb{B}_1 \rightarrow \mathbb{R}$ satisfying*

$$\| (D_m)_a f(x_1, \cdots, x_n) \| \leq \varphi(x_1, \cdots, x_n) , \tag{5.2}$$

for all $a \in B_1$, $x_1, \cdots, x_n \in_B \mathbb{B}_1 \backslash \{0\}$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in_B \mathbb{B}_1$, then there exists a unique $n-$dimensional $B_{sa}-$quadratic mapping $Q :_B \mathbb{B}_1 \rightarrow_B \mathbb{B}_2$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{4\, {}_{n-3}C_{m-2}} \widetilde{\varphi}(x, -x, x, 0, \cdots, 0) , \tag{5.3}$$

for all $x \in_B \mathbb{B}_1$.

Proof. By the same reasoning as the proof of Theorem 3.1, it follows from the inequality of the statement $a = 1$ that there exists a unique $n-$dimensional quadratic mapping $Q :_B \mathbb{B}_1 \rightarrow_B \mathbb{B}_2$ defined by

$$Q(x) = \lim_{m \to \infty} 2^{-2m} f(2^m x) ,$$

which satisfies the inequality (5.3) for all $x \in_B \mathbb{B}_1$. Under the assumptions that either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in_B \mathbb{B}_1$, by the same reasoning as the proof of [3], one can show that Q is $\mathbb{R}$-quadratic, that is, $Q(tx) = t^2 Q(x)$ for all $t \in \mathbb{R}$, for all $x \in_B \mathbb{B}_1$.

Putting $x_1 = 2^{\, m-1} x$ and $x_j = 0\, (j = 2, \cdots, n)$ in (5.2) and dividing the resulting inequality by 2^{2m},

$$\begin{aligned} &\frac{1}{2^{2m}} \| f(a 2^{m-1} x) - b f(2^{m-1} x) \| \\ &\leq \frac{1}{2^{2m}} \frac{1}{{}_{n-1}C_{m-1}} \phi(2^{m-1} x, 0, \cdots, 0) , \end{aligned}$$

for all $x_1, \cdots, x_n \in_B \mathbb{B}_1$. By the definition of Q,

$$Q(ax) = \lim_{s \to \infty} \frac{1}{2^{2s}} f(2^s a x) = \lim_{s \to \infty} \frac{2^2}{2^{2s}} b f(2^{s-1} x) = b Q(x).$$

for every $x \in_B \mathbb{B}_1$, for every $a \in B(|a| = 1)$. For $a \in B \setminus \{0\}$,

$$Q(ax) = Q(|a| \frac{a}{|a|} x) = |a|^2 Q(\frac{a}{|a|} x) = |a|^2 \frac{b}{|a|^2} Q(x) = b Q(x),$$

for all $x \in_B \mathbb{B}_1$. Thus Q is $n-$dimensional $B_{sa}-$quadratic, which completes the proof. ∎

Remark 5.4. By the similar method, we also obtain the unique $n-$dimensional $B-$quadratic mapping on the same conditions.

Corollary 5.5. *Let $f :_B \mathbb{B}_1 \rightarrow_B \mathbb{B}_2$ be a mapping with $f(0) = 0$ for the case (5.1) which there exists mapping $\varphi :_B \mathbb{B}_1 \rightarrow \mathbb{R}$ satisfying*

$$\begin{aligned}&\| b \cdot_{n-2} C_{m-2} f(\sum_{j=1}^{n} x_j) + b \cdot {}_{n-2}C_{m-1} \sum_{i=1}^{n} f(x_i) \\ &\quad - \sum_{1 \le i_1 < \cdots < i_m \le n} f(a(x_{i_1} + \cdots + x_{i_m})) \ \| \\ &\le \varphi(x_1, \cdots, x_n),\end{aligned}$$

for all $a \in B_1$, for all $x_1, \cdots, x_n \in_B \mathbb{B}_1 \backslash \{0\}$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in_B \mathbb{B}_1$, then there is an unique $n-$dimensional $B_{sa}-$quadratic mapping $Q :_B \mathbb{B}_1 \rightarrow_B \mathbb{B}_2$ which satisfies the inequality (5.3) for all $x \in_B \mathbb{B}_1$.

Proof. By the similar method of the proof of Theorem 5.3, one can obtain the result. ∎

Definition 5.6. An $n-$dimensional quadratic mapping $Q : \mathbb{B} \rightarrow B$ is called an *$n-$dimensional A-quadratic* mapping if $Q(ax) = aQ(x)a^*$ for all $a \in B, x \in \mathbb{B}$.

Theorem 5.7. *Let $f :_B \mathbb{B}_1 \rightarrow_B \mathbb{B}_2$ be a mapping with $f(0) = 0$ for the cases (5.1) and (5.2) and define $Q :_B \mathbb{B}_1 \rightarrow_B \mathbb{B}_2$ defined by for all $x \in_B \mathbb{B}_1$,*

$$Q(x) = \lim_{m\to\infty} 2^{-2m} f(2^m x),$$

which there is mapping $\psi :_B \mathbb{B}_1 \rightarrow \mathbb{R}$ satisfying

$$\| Q(ax) - aQ(x)a^* \| \le \psi(x) \ and \ \lim_{m\to\infty} \frac{\psi(2^{2m}x)}{2^{2m}} = 0 \tag{5.4}$$

for all $a \in B_1$, $x \in_B \mathbb{B}_1$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in_B \mathbb{B}_1$, then Q is the unique $n-$dimensional $A-$quadratic mapping which satisfies the inequality (5.3) for all $x \in_B \mathbb{B}_1$.

Proof. By the same reasoning as the proof of Theorem 3.1, Q is well-defined and Q is the unique $n-$dimensional $\mathbb{R}-$quadratic mapping which satisfies the inequality (5.3) for all $x \in_B \mathbb{B}_1$. By (5.4), for each element $a \in B_1$, $x \in_B \mathbb{B}_1$,

$$Q(ax) = aQ(x)a^*.$$

Since Q is $n-$dimensional $\mathbb{R}$-quadratic,

$$Q(ax) = Q(|a|\frac{a}{|a|}x) = |a|^2 Q(\frac{a}{|a|}x) = |a|^2 \frac{a}{|a|} Q(x) \frac{a^*}{|a|} = aQ(x)a^*,$$

for all $a \in B(|a| \neq 0), x \in_B \mathbb{B}_1$. Thus Q is $n-$dimensional $A-$quadratic, as desired. ∎

6 Stability using alternative fixed point

In this section, we will investigate the stability of the given $n-$dimensional quadratic functional equation (3.1) using the alternative fixed point. Before proceeding the proof, we will state theorem, the alternative of fixed point.

Theorem 6.1 (The alternative of fixed point [13], [22]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^n x,\, T^{n+1}x) = \infty \textit{ for all } n \geq 0\,,$$

or there exists a natural number n_0 such that

1. $d(T^n x,\, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
2. *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
3. *y^* is the unique fixed point of T in the set*
$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\}\,;$$
4. $d(y, y^*) \leq \frac{1}{1-L}\, d(y, Ty)$ *for all* $y \in \triangle$.

Now, let $\phi : X^n \to [0, \infty)$ be a function such that

$$\lim_{m\to\infty} \frac{\phi(\lambda_i^m x_1, \cdots, \lambda_i^m x_n)}{\lambda_i^{2m}} = 0\,,$$

for all $x_1, \cdots, x_n \in X$, where $\lambda_i = 2$ if $i = 0$ and $\lambda_i = \frac{1}{2}$ if $i = 1$.

Theorem 6.2. *Let $2 \leq m \leq n-1$ be an integer number. Suppose that an even function $f : X \to Y$ satisfies the functional inequality*

$$\| D_m f(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n)\,, \tag{6.1}$$

for all $x_1, \cdots, x_n \in X$ and $f(0) = 0$. If there exists $L = L(i) < 1$ such that the function

$$x \mapsto \psi(x) = \phi(\frac{1}{2}x, -\frac{1}{2}x, \frac{1}{2}x, 0, \cdots, 0) \tag{6.2}$$

has the property

$$\psi(x) \leq L \cdot \lambda_i^2 \cdot \psi(\frac{x}{\lambda_i})\,, \tag{6.3}$$

for all $x \in X$, then there exists a unique $n-$dimensional quadratic function $Q : X \to Y$ such that the inequality

$$\| f(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L}\, \psi(x) \tag{6.4}$$

holds for all $x \in X$.

Proof. Consider the set

$$\Omega = \{g | g : X \to Y\,, g(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(g,\, h) = d_\psi(g, h) = inf\{K \in (0, \infty) |\ \ \| g(x) - h(x) \| \leq K\psi(x)\,, x \in X\}\,.$$

It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\lambda_i^2}\, g(\lambda_i x)\,,$$

for all $x \in X$. Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g, h) < K \ \ &\Rightarrow \ \ \| g(x) - h(x) \| \leq K\psi(x)\,, \text{ for all } x \in X\,, \\ &\Rightarrow \ \ \| \frac{1}{\lambda_i^2} g(\lambda_i x) - \frac{1}{\lambda_i^2} h(\lambda_i x) \| \leq \frac{1}{\lambda_i^2} K\psi(\lambda_i x)\,, \text{ for all } x \in X\,, \\ &\Rightarrow \ \ \| \frac{1}{\lambda_i^2} g(\lambda_i x) - \frac{1}{\lambda_i^2} h(\lambda_i x) \| \leq LK\psi(x)\,, \text{ for all } x \in X\,, \\ &\Rightarrow \ \ d(Tg,\, Th) \leq LK\,. \end{aligned}$$

Hence we have that

$$d(Tg,\, Th) \leq L\, d(g,\, h)\,,$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L. By setting $x_1 = x\,, x_2 = -x\,, x_3 = x$, and $x_4 = \cdots = x_n = 0$, we have the inequality (3.5) as in the proof of Theorem 3.1 and we use the inequality (6.3) with the case where $i = 0$, which is reduced to

$$\| f(x) - \frac{1}{4} f(2x) \| \leq \frac{1}{4 \cdot_{n-3} C_{m-2}} \psi(2x) \leq L\, \psi(x)\,, \tag{6.5}$$

for all $x \in X$, that is, $d(f, Tf) \leq L = L^1 < \infty$. Now, replacing x by $\frac{1}{2}x$ in the inequality (6.5), multiplying 4, and using the inequality (6.3) with the case where $i = 1$, we have that

$$\| f(x) - 2^2 f(\frac{x}{2}) \| \leq \psi(x)\,,$$

for all $x \in X$, that is, $d(f, Tf) \leq 1 = L^0 < \infty$. In both cases we can apply the fixed point alternative and since $\lim_{r\to\infty} d(T^r f, Q) = 0$, there exists a fixed point Q of T in Ω such that

$$Q(x) = \lim_{n\to\infty} \frac{f(\lambda_i^r x)}{\lambda_i^{2r}}\,, \tag{6.6}$$

for all $x \in X$. Letting $x_j = \lambda_i^r x_j$ for $j = 1, \cdots, n$ in the inequality (6.1) and dividing by λ_i^{2r},

$$\begin{aligned} \| D_m Q(x, \cdots, x_n) \| \ \ &= \ \ \lim_{r\to\infty} \frac{\| D_m f(\lambda_i^r x_1, \cdots, \lambda_i^r x_n) \|}{\lambda_i^{2r} x_1} \\ &\leq \ \ \lim_{r\to\infty} \frac{\| \phi(\lambda_i^r x_1, \cdots, \lambda_i^r x_n) \|}{\lambda_i^{2r} x_1} = 0\,, \end{aligned}$$

for all $x_1, \cdots, x_n \in X$; that is it satisfies the inequality (2.1). By Lemma 2.1, the Q is quadratic. Also, the fixed point alternative guarantees that such a Q is the unique function such that

$$\| f(x) - Q(x) \| \leq K \, \psi(x) \, ,$$

for all $x \in X$ and some $K > 0$. Again using the fixed point alternative, we have

$$d(f, Q) \leq \frac{1}{1-L} d(f, Tf) \, .$$

Hence we may conclude that

$$d(f, Q) \leq \frac{L^{1-i}}{1-L} \, ,$$

which implies the inequality (6.4). ■

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