# Disjointness preserving Fredholm operators in ultrametric spaces of continuous functions

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**Abstract**

We give a complete description of the Fredholm disjointness preserving operators between ultrametric spaces of (bounded and not necessarily bounded) continuous functions defined on $\mathbb{N}$-compact spaces.

## 1 Introduction

The aim of this paper is to provide a complete description of Fredholm disjointness preserving maps between some spaces of continuous functions in the ultrametric context (roughly speaking, a disjointness preserving map is a map which preserves zero products. See definitions below). We will study the case of operators defined on spaces $C(X)$ (of all continuous functions) and $C^*(X)$ (of all *bounded* continuous functions).

A study with a similar purpose has recently been carried out in the real and complex settings (see [7]), although the techniques we use in this paper are independent of those used there.

The spaces studied in [7] are those of functions vanishing at infinity, requiring in particular the underlying topological spaces to be locally compact. On the other hand, a representation of bijective disjointness preserving maps defined between this kind of spaces was previously known (see [5, 6, 8]).

In the ultrametric context we know the representation of disjointness preserving maps defined on spaces $C(X)$ or $C^*(X)$, when the underlying topological spaces are $\mathbb{N}$-compact (see [1, 2]). This will be useful in our study.

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We also wish to point out that some aspects of Fredholm operators in the ultrametric setting have been recently studied (see for instance [3, 9, 11]), but they are not directly related to our results here.

### 1.1 Basic definitions

We start by defining disjointness preserving maps (also known as separating) between rings, and Fredholm operators between vector spaces.

**Definition 1.1.** *Let $\mathfrak{R}, \mathfrak{R}'$ be rings. A map $T : \mathfrak{R} \to \mathfrak{R}'$ is said to be disjointness preserving if $(Ta)(Tb) = 0$ whenever $ab = 0$, $a, b \in \mathfrak{R}$.*

**Definition 1.2.** *Let $V$ and $W$ be linear spaces over a field. A linear operator $T : V \to W$ is said to be Fredholm if its kernel,* $\operatorname{Ker} T$*, and the codimension of its range,* $\operatorname{codim} T := W/R(T)$*, are finite.*

### 1.2 Notation

Let $\mathbb{K}$ be a field endowed with a nonarchimedean valuation, and let $X$ be a topological space. Then $C(X)$ will denote the space of all $\mathbb{K}$-valued continuous functions on $X$, and $C^*(X)$ will be the space of all bounded $\mathbb{K}$-valued continuous functions on $X$.

In general we will consider $C(X)$ and $C^*(X)$ just as linear spaces over $\mathbb{K}$, with no additional topological structure. Nevertheless, the sup norm, given as $\|f\| := \sup_{x \in X} |f(x)|$ for each $f \in C^*(X)$ makes the space into a Banach space. We will at some points use the notation $\|f\|$ to denote the supremum of absolute values taken by $f$.

For a (not necessarily continuous) $f : X \to \mathbb{K}$, we denote by $c(f)$ and $z(f)$ its cozero and zero sets respectively, that is, $c(f) := \{x \in X : f(x) \neq 0\}$, and $z(f) := X \setminus c(f)$. For a subset $Z$ of $X$, $\operatorname{cl}_X Z$ will be the closure of $Z$ in $X$.

Given a set $A$, we denote by $\xi_A$ the $\mathbb{K}$-valued characteristic function on $A$. Also $B_{\mathbb{K}}(0, 1)$ will be the closed unit ball with center 0 in $\mathbb{K}$.

For all basic and unexplained terminology we refer the reader to [10].

Notice that if $\mathbb{K}$ is locally compact and $X$ is $\mathbb{N}$-compact, then every bounded continuous function $f : X \to \mathbb{K}$ can be extended to a continuous function $f' : \beta_0 X \to \mathbb{K}$ (where $\beta_0 X$ denotes the Banaschewski compactification of $X$). This implies that $C^*(X)$ and $C(\beta_0 X)$ are indistinguishable both as rings and as linear spaces. This fact will be important when choosing the contexts we will work in.

**Assumptions on underlying spaces and on fields.** From now on, unless otherwise stated, the topological spaces $X$ and $Y$ are assumed to be $\mathbb{N}$-compact.

$\mathbb{K}$ will be a field endowed with a nonarchimedean valuation for which it is complete. In the case of spaces $C(X)$, $C(Y)$, no extra assumptions will be made on $\mathbb{K}$. Nevertheless, if $X$ (resp. $Y$) is *not* compact and we deal with the space $C^*(X)$ (resp. $C^*(Y)$), we will also assume that $\mathbb{K}$ is *not* locally compact.

**Statement of results.** Some results will be valid both for spaces of continuous and bounded continuous functions.

We will use a special notation: Throughout $\mathfrak{A}(X)$ and $\mathfrak{A}(Y)$ will be some subalgebras of $C(X)$ and $C(Y)$, respectively. As it will be announced each time, when in a statement we say that $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ or $(C(X), C(Y))$, we mean that this statement is true in the following two cases:

- when $\mathfrak{A}(X) = C^*(X)$ and $\mathfrak{A}(Y) = C^*(Y)$,
- when $\mathfrak{A}(X) = C(X)$ and $\mathfrak{A}(Y) = C(Y)$.

This means in particular the sentence *Let* $T : \mathfrak{A}(X) \to \mathfrak{A}(Y)$ *be... cannot* be translated into *Let* $T : C^*(X) \to C(Y)$ *be...*

We suppose that $T : \mathfrak{A}(X) \to \mathfrak{A}(Y)$ is any (fixed) disjointness preserving Fredholm operator. We define $D := \bigcup_{f \in \mathfrak{A}(X)} c(Tf)$, that is, $Y \setminus D$ consists of those points in $Y$ whose image by $Tf$ is equal to 0 for every $f \in \mathfrak{A}(X)$.

It is well known that in some contexts, every disjointness preserving operator has an associated map called support map. The idea is as follows: given any point $y \in D$, there exists a point $x$ in $X$ (or in a certain compactification of $X$) with the property that for every neighborhood $U$ of $x$ (in $X$ or in that compactification), there exists $f \in \mathfrak{A}(X)$ such that $c(f) \subset U$ and $(Tf)(y) \neq 0$. The point $x$ is usually called support point of $y$, and the support map is that sending each point of $D$ to its support point.

In our case, if $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ or $(C(X), C(Y))$, the support map of $T$ is a function $h : D \to \beta_0 X$ (see [1, 2]).

Among the properties of the support map we have the following, which will be used later.

**Proposition 1.1.** *(see [1, 2]) Let* $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ *or* $(C(X), C(Y))$. *The support map* $h : D \to \beta_0 X$ *is continuous. Also, if* $T$ *is injective, then its image is dense in* $\beta_0 X$. *Moreover, for* $y \in D$ *and* $U \subset \beta_0 X$, *if* $h(y) \notin \mathrm{cl}_{\beta_0 X} U$, *then* $(Tf)(y) = 0$ *for every* $f \in \mathfrak{A}(X)$ *such that* $c(f) \subset U$.

Now, we may split $D$ into three different subsets, namely

$$\begin{aligned} D_1 &:= \{y \in D : h(y) \in \beta_0 X \setminus X\}, \\ D_2 &:= \{y \in D \setminus D_1 : \exists f \in \mathfrak{A}(X) \text{ such that } f(h(y)) = 0 \text{ and } (Tf)(y) \neq 0\}, \text{ and} \\ D_3 &:= D \setminus (D_1 \cup D_2). \end{aligned}$$

## 2 Main results

Here we state the main results of the paper.

**Theorem 2.1.** *Suppose that* $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$. *Then*

(1) *If* $N := \dim \operatorname{Ker} T$, *then there exists a set* $A$ *consisting of* $N$ *isolated points in* $X$ *such that* $\operatorname{Ker} T = \{f : f(X \setminus A) \equiv 0\}$.

(2) *The sets* $D_1, D_2$ *and* $Y \setminus D$ *are finite and consist of isolated points.*

(3) *The image of the restriction of $h$ to $D_3$ is $X \setminus A$. Moreover the preimage in $D_3$ by $h$ of each point of $X \setminus A$ is a finite set, which consists of a single point in all but a finite number of cases.*

(4) *If in $D_3$, we consider the equivalence relation $xRy$ if $h(x) = h(y)$, then the map $h_R : D_3/R \to X \setminus A$ (sending the equivalence class of each $y \in D_3$ into $h(y)$) is a surjective homeomorphism.*

(5) *If $M := \operatorname{codim} R(T)$, then*

$$\begin{aligned} M &= \text{card } (Y \setminus D) + \text{card } D_1 + \text{card } D_2 \\ &+ \sum_{x \in X \setminus A} \left[\text{card } \left(D_3 \cap h^{-1}(\{x\})\right) - 1\right]. \end{aligned}$$

(6) *There exists $a \in C^*(D_3)$ such that $\inf\{|a(y)| : y \in D_3\} > 0$ and such that $(Tf)(y) = a(y)f(h(y))$ for every $f \in C^*(X)$ and $y \in D_3$.*

**Theorem 2.2.** *Suppose that $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C(X), C(Y))$. Then* (1), (2), (3), (4), and (5) in Theorem 2.1 hold. On the other hand, (6) must be replaced by the following:

(6′) There exists $a \in C(D_3)$ such that $a(y) \neq 0$ for every $y \in D_3$, and such that $(Tf)(y) = a(y)f(h(y))$ for every $f \in C(X)$ and $y \in D_3$.

**Remark.** We see in Theorem 2.1 (and 2.2) that all points in $D_1 \cup D_2 \cup (Y \setminus D)$ are isolated. One may wonder if when $x \in X \setminus A$ satisfies that $D_3 \cap h^{-1}(\{x\})$ has more than one point, each point of the subset must be isolated. In fact this is not true in general. It could even be the case that none of the points of the subset is isolated, as we see in the following example.

**Example.** Let $X := \mathbb{Z}_p$, $Y := \mathbb{Z}_p \times \{0, 1\}$. Take any $(x, i) \in Y$ with $x \neq 0$, and suppose that $|x|_p = p^{-n}$, $n \in \mathbb{N} \cup \{0\}$. Then define $h(x, i) := p^n x$ if $i = 0$, and $h(x, i) := p^{n+1}x$ if $i = 1$. Define also $h(0, i) := 0$ for $i = 1, 2$. It is easy to see that the map $h : Y \to X$ is continuous and the range of the composition map $T : C(X) \to C(Y)$ defined as $Tf := f \circ h$ has codimension 1. Nevertheless $D_3 = Y$ has no isolated points.

## 3 Some lemmas and propositions

**Lemma 3.1.** *Let $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ or $(C(X), C(Y))$. Let $f \in \mathfrak{A}(X)$ and $y \in Y$ be such that $(Tf)(y) = 0$. If $U \subset X$ is clopen, then $(Tf\xi_U)(y) = 0$.*

*Proof.* It is clear that if for some clopen $U$, we have $(Tf\xi_U)(y) \neq 0$, then the fact $(Tf)(y) = 0$ would imply that $(Tf\xi_{X \setminus U})(y) = -(Tf\xi_U)(y) \neq 0$, and this would go against the fact that $T$ is disjointness preserving (notice that we do not use the fact that $T$ is Fredholm, but only linearity). ∎

**Proposition 3.2.** *Let $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ or $(C(X), C(Y))$, and let $N := \dim \operatorname{Ker} T$. Then there exists a subset $A$ of $X$ consisting of $N$ isolated points such that $\operatorname{Ker} T = \{f \in \mathfrak{A}(X) : f(X \setminus A) \equiv 0\}$.*

*Proof.* Let
$$A := \bigcup_{f \in \operatorname{Ker} T} c(f).$$
We are going to see that card $A = N$. Suppose that card $A > N$. Then we can take mutually distinct $x_1, \ldots, x_{N+1} \in A$, and functions $f_1, \ldots, f_{N+1} \in \operatorname{Ker} T$ satisfying $x_i \in c(f_i)$ for each $i$.

Next take pairwise disjoint clopen subsets $U_1, \ldots, U_{N+1} \subset A$ with $x_i \in U_i$ for every $i$, and define $g_i := f_i \xi_{U_i}$, $i = 1, \ldots, N+1$. Now by Lemma 3.1, each $g_i$ belongs to $\operatorname{Ker} T$. This implies that $\dim \operatorname{Ker} T \geq N + 1$, which is impossible. We deduce that $A$ consists of at most $N$ points and, since it is open, they must be isolated. Finally, is is obvious that if $A$ does not contain $N$ points, then $\dim \operatorname{Ker} T < N$. ∎

**Lemma 3.3.** *Let $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ or $(C(X), C(Y))$. Let $(f_n)$ be a sequence in $\mathfrak{A}(X)$ such that $c(f_n) \cap c(f_m) = \emptyset$ if $n \neq m$. Suppose that the map $f : X \to \mathbb{K}$, defined pointwise as $f(x) := \sum_{n=1}^{\infty} f_n(x)$ for each $x \in X$, satisfies $f \in \mathfrak{A}(X)$.*

*Then there exists a (not necessarily continuous) function $g : Y \to \mathbb{K}$ satisfying $c(g) \cap c(Tf_n) = \emptyset$ for every $n \in \mathbb{N}$, and $Tf = g + \sum_{n=1}^{\infty} Tf_n$ (pointwise).*

*Proof.* Let $D_0 := \bigcup_{n=1}^{\infty} c(Tf_n)$. Given any $y \in D_0$, there exists $n_y \in \mathbb{N}$ such that $(Tf_{n_y})(y) \neq 0$. Also, since $T$ is disjointness preserving and $c(f_{n_y}) \cap c\left(\sum_{n \neq n_y} f_n\right) = \emptyset$, then we see that $\left(T \sum_{n \neq n_y} f_n\right)(y) = 0$, and consequently $(Tf)(y) = (Tf_{n_y})(y) + \left(T \sum_{n \neq n_y} f_n\right)(y) = (Tf_{n_y})(y) \neq 0$. So we conclude that for each $y \in D_0$, $(Tf)(y) = \sum_{n=1}^{\infty} (Tf_n)(y)$.

It is now clear that, if we define $g := \xi_{Y \setminus D_0} Tf$, then $Tf = g + \sum_{n=1}^{\infty} Tf_n$ (pointwise), as we wanted to see. ∎

**Lemma 3.4.** *Let $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ or $(C(X), C(Y))$. Let $(f_n)$ be a sequence in $\mathfrak{A}(X)$ such that $c(f_n) \cap c(f_m) = \emptyset$ if $n \neq m$. Suppose that the map $f : X \to \mathbb{K}$, defined pointwise as $f(x) := \sum_{n=1}^{\infty} f_n(x)$ for each $x \in X$, satisfies $f \in \mathfrak{A}(X)$.*

*Assume that at least one of the following two conditions holds:*

(1) $\lim_{n \to \infty} \|f_n\| = 0$,

(2) *$X$ is not compact, and $\bigcap_{k=1}^{\infty} \left(\operatorname{cl}_X \bigcup_{n=k}^{\infty} c(f_n)\right) = \emptyset$.*

*Then $\sum_{n=1}^{\infty} Tf_n$ belongs to $\mathfrak{A}(Y)$.*

*Proof.* As in the proof of Lemma 3.3, let $D_0 := \bigcup_{n=1}^{\infty} c(Tf_n)$. Lemma 3.3 gives us the representation $Tf = g + \sum_{n=1}^{\infty} Tf_n$, with $c(g) \cap c(Tf_n) = \emptyset$ for every $n \in \mathbb{N}$. In particular, this implies that, since $Tf = \sum_{n=1}^{\infty} Tf_n$ on the open set $D_0$ and $Tf$ belongs to $\mathfrak{A}(Y)$, then $\sum_{n=1}^{\infty} Tf_n$ is continuous on $D_0$ (and bounded when $\mathfrak{A}(Y) = C^*(Y)$). On the other hand, since $\sum_{n=1}^{\infty} Tf_n \equiv 0$ on $Y \setminus \operatorname{cl}_Y D_0$, then we have that $\sum_{n=1}^{\infty} Tf_n$ belongs to $\mathfrak{A}(Y)$ if and only if it is continuous at every point of $\partial D_0$, the boundary of $D_0$.

Next suppose that for a certain point $y \in \partial D_0$ there exists $k \in \mathbb{N}$ such that $y$ does not belong to the closure of $\bigcup_{n=k+1}^{\infty} c(Tf_n)$. Then we have that $T\left(\sum_{n=k+1}^{\infty} f_n\right) = T\left(f - \sum_{n=1}^{k} f_n\right) = g + \sum_{n=k+1}^{\infty} Tf_n$. Now, the fact that $y \notin \mathrm{cl}_Y \bigcup_{n=k+1}^{\infty} c(Tf_n)$ implies that $\sum_{n=k+1}^{\infty} Tf_n$ is continuous at $y$, that is, $\sum_{n=1}^{\infty} Tf_n$ is continuous at $y$.

As a consequence we see that the points where $\sum_{n=1}^{\infty} Tf_n$ may not be continuous are included in the set

$$\partial_0 D_0 := \left\{ y \in \partial D_0 : y \in \mathrm{cl}_Y \bigcup_{n=k}^{\infty} c(Tf_n) \ \forall k \in \mathbb{N} \right\}.$$

Suppose then that $y_0 \in \partial_0 D_0$, and that $\sum_{n=1}^{\infty} Tf_n$ is not continuous at $y_0$. This implies that $(Tf)(y_0) \neq 0$. Put $r := |(Tf)(y_0)|$. Let $U_0$ be a clopen neighborhood of $y_0$ in $Y$ such that $|(Tf)(y) - (Tf)(y_0)| < r/2$ whenever $y \in U_0$.

Assume first (1), that is, $\lim_{n\to\infty} \|f_n\| = 0$, and take a sequence $(\alpha_n)$ in $\mathbb{K}$ such that $\lim_{n\to\infty} |\alpha_n| = +\infty$ and $\lim_{n\to\infty} |\alpha_n| \, \|f_n\| = 0$. Notice that a sequence like this exists, because it is enough to take $\alpha_n \in \mathbb{K}$ satisfying $|\alpha_n| \leq 1/\sqrt{\|f_n\|}$ for each $n \in \mathbb{N}$. Then define $g' := \sum_{n=1}^{\infty} \alpha_n f_n$, which is clearly continuous and bounded. By continuity of the map $Tg'$, there exists a clopen neighborhood $U_1$ of $y_0$ such that

$$|(Tg')(y) - (Tg')(y_0)| < r/2 \tag{1}$$

for every $y \in U_1$. Since we are assuming that $y_0$ belongs to $\partial_0 D_0$, then there is a strictly increasing sequence $(n_k)$ of natural numbers such that $U_0 \cap U_1 \cap c(Tf_{n_k}) \neq \emptyset$ for every $n_k$. For each $n_k$, take a point $y_{n_k}$ in that intersection. We have that $|(Tf)(y_{n_k}) - (Tf)(y_0)| < r/2$, so $|(Tf)(y_{n_k})| = r$ for every $n_k$, which is to say that $|(Tf_{n_k})(y_{n_k})| = r$ for every $n_k$. Consequently $|\alpha_{n_k}(Tf)(y_{n_k})| = |\alpha_{n_k}| r$, and this implies that $|(Tg')(y_{n_k})| = |\alpha_{n_k}| r$, which gives that $Tg'$ is not bounded in $U_1$, against Equation 1. So, when we assume $\lim_{n\to\infty} \|f_n\| = 0$, the function $\sum_{n=1}^{\infty} Tf_n$ is continuous.

Assume next (2). Then we can take a sequence $(\alpha_n)$ in $\mathbb{K}$ such that $|\alpha_n - \alpha_m| > 1/2$ whenever $n \neq m$. Notice that, if $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$, or more in general if $\mathbb{K}$ is not locally compact, then we can take it such that $1/2 < |\alpha_n| \leq 1$ for every $n \in \mathbb{N}$.

Now define $g' := \sum_{n=1}^{\infty} \alpha_n f_n$, which is clearly an element of $\mathfrak{A}(X)$, because we are assuming (2). It is easy to see that $\alpha_n f \equiv g'$ on $c(f_n)$, which means that $\alpha_n Tf \equiv Tg$ on $c(Tf_n)$. Now we have that there exists a neighborhood $U_1 \subset U_0$ of $y_0$ such that $|(Tg')(y) - (Tg')(y_0)| < r/2$ whenever $y \in U_1$. Take $k_1, k_2 \in \mathbb{N}$ such that $U_1 \cap c(Tf_{k_i}) \neq \emptyset$ for $i = 1, 2$. Then we have that, for $i = 1, 2$, if $y_i \in U_1 \cap c(Tf_{k_i})$, $|(Tf)(y_1) - (Tf)(y_2)| < r/2$ and $|(Tg)(y_1) - (Tg)(y_2)| < r/2$, which implies

$$|\alpha_{k_1}(Tf_{k_1})(y_1) - \alpha_{k_1}(Tf_{k_2})(y_2)| < r/2$$

and

$$|\alpha_{k_1}(Tf_{k_1})(y_1) - \alpha_{k_2}(Tf_{k_2})(y_2)| < r/2.$$

This would imply that

$$|\alpha_{k_1}(Tf_{k_2})(y_2) - \alpha_{k_2}(Tf_{k_2})(y_2)| < r/2,$$

so $|\alpha_{k_1} - \alpha_{k_2}| < 1/2$, which goes against the way we have taken the sequence $(\alpha_n)$. ∎

**Proposition 3.5.** *Let* $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ *or* $(C(X), C(Y))$*. Then* $D_1$ *consists of a finite set of isolated points.*

*Proof.* It is obvious that if $X$ is compact, then $D_1$ is empty. Assume then that $X$ is not compact. Let $L \in \mathbb{N}$, and suppose that $y_1, \dots, y_L$ are points in $D_1$. We are going to find an open set contained in $D_1$ and which contains all $y_i$.

First, it is clear that there exist $K \in \mathbb{N}$, $K \leq L$, and $x_1, \dots, x_K \in \beta_0 X \setminus X$, such that $h(\{y_1, \dots, y_L\}) = \{x_1, \dots, x_K\}$.

Since $X$ is $\mathbb{N}$-compact, we know that there exists a sequence $(U_n)$ of clopen sets of $\beta_0 X$ all of them containing $\{x_1, \dots, x_K\}$ such that $U_{n+1} \subset U_n$ for every $n \in \mathbb{N}$ and such that $X \cap (\bigcap_{n=1}^{\infty} U_n) = \emptyset$. Without loss of generality we may assume $U_1 = \beta_0 X$. For each $n \in \mathbb{N}$, let $V_n := U_n \setminus U_{n+1}$, and let $f_n$ be the restriction to $X$ of the characteristic function on $V_n$.

On the other hand, it is easy to see that there exists $f \in \mathfrak{A}(X)$ such that $(Tf)(y_i) \neq 0$ for each $i \in \{1, \dots, L\}$. Also $f = \sum_{n=1}^{\infty} f f_n$, so by Lemmas 3.3 and 3.4(2), we have that there exists $g_1 \in \mathfrak{A}(Y)$ with $c(g_1) \cap c(T f f_n) = \emptyset$ for every $n \in \mathbb{N}$, and such that $Tf = g_1 + \sum_{n=1}^{\infty} T f f_n$.

Now fix any $k \in \mathbb{N}$. It is clear that $T\left(\sum_{n=k+1}^{\infty} f f_n\right) = g_1 + \sum_{n=k+1}^{\infty} T f f_n$. Consequently, given $y \in c(g_1)$, we have that $h(y)$ does not belong to $V_1 \cup \dots \cup V_k$, that is, $h(y)$ belongs to $U_{k+1}$. Then we have $h(c(g_1)) \subset \bigcap_{n=1}^{\infty} U_n$. Also, it is easy to see that $y_1, \dots, y_L \in c(g_1)$.

Let $M := \operatorname{codim} R(T)$. We are going to see that the open set $c(g_1)$ cannot have more than $M$ elements. Otherwise we can find $M+1$ pairwise disjoint (nonempty) clopen subsets $W_1, \dots, W_{M+1}$ of $c(g_1)$. Since the codimension of $R(T)$ is $M$, then there exist $\alpha_1, \alpha_2, \dots, \alpha_{M+1} \in \mathbb{K}$ such that $g_2 := \sum_{i=1}^{M+1} \alpha_i \xi_{W_i}$ belongs to $R(T) \setminus \{0\}$.

Let $f' \in \mathfrak{A}(X)$ such that $Tf' = g_2$. Let us see that $f' \equiv 0$. Notice first that, for any $k \in \mathbb{N}$, $c(f') \cap c\left(\sum_{n=k}^{\infty} f f_n\right) \neq \emptyset$ because, otherwise, due to the disjointness preserving property of $T$, we would have $c(g_2) \cap c\left(T \sum_{n=k}^{\infty} f f_n\right) = \emptyset$.

We deduce that there are infinitely many $n \in \mathbb{N}$ such that $c(f') \cap c(f_n) \neq \emptyset$. By Proposition 3.2, there exists $n_0 \in \mathbb{N}$ such that $c(f') \cap c(f_{n_0}) \neq \emptyset$ and $Tl \neq 0$ for every $l \in \mathfrak{A}(X)$ with $c(l) \subset c(f_{n_0})$. Take $U$ clopen and nonempty with $U \subset c(f') \cap c(f_{n_0})$. We know then that there exists a point $y \in Y$ such that $(T(f' \xi_U))(y) \neq 0$, which by Lemma 3.1 implies that $(Tf')(y) \neq 0$, that is, $g_2(y) \neq 0$. On the other hand, since $U \subset c(f_{n_0})$, we have that $\left(T \sum_{n=n_0+1}^{\infty} f f_n\right)(y) = 0$, which implies that $g_1(y) = 0$, and consequently $g_2(y) = 0$.

This contradiction proves that, for $L \in \mathbb{N}$, the open set $c(g_1)$ has at most $M$ points. This implies that there is a finite number of them, $L \leq M$, and that all of them are isolated. ∎

**Proposition 3.6.** *Let* $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ *or* $(C(X), C(Y))$*. Then* $D_2$ *consists of a finite set of isolated points.*

*Proof.* The proof of this proposition is similar to that of Proposition 3.5. Let $L$, $K$, and $M$ be defined as there. Suppose that $y_1, \dots, y_L$ are points in $D_2$ such that $h(\{y_1, \dots, y_L\}) = \{x_1, \dots, x_K\} \subset X$. Next consider pairwise disjoint open subsets $U_1, \dots, U_K$ of $X$ such that $x_i \in U_i$ for each $i$, and $h(D_1) \cap \operatorname{cl}_{\beta_0 X}(U_1 \cup \dots \cup U_K) = \emptyset$. For each $i = 1, \dots, K$, take functions $f_i \in \mathfrak{A}(X)$ such that we have $c(f_i) \subset U_i$,

$\|f_i\| = 1$, $f_i(x_i) = 0$, and $(Tf_i)(y) \neq 0$ whenever $h(y) = x_i$ and $y \in \{y_1, \ldots, y_L\}$. Let $f := \sum_{i=1}^{K} f_i$.

For each $n \in \mathbb{N}$, let $A_n := \{x \in X : |f(x)| \leq 1/n\}$. We can write $f = \sum_{n=1}^{\infty} f\xi_{A_n}$. By Lemmas 3.3 y 3.4(1), we have $Tf = g_1 + g_2$, where $g_1 := \sum_{n=1}^{\infty} T(f\xi_{A_n})$, $g_2 \in \mathfrak{A}(Y)$, and $c(g_1) \cap c(g_2) = \emptyset$.

Now it is clear from Proposition 1.1 that $h(c(g_1)) \subset c(f)$ and $h(c(g_2)) \subset z(f)$. Also $\{y_1, \ldots, y_L\} \subset c(g_2)$, so we are in a similar situation as in the proof of Proposition 3.5. That is, for any $L$ points in $D_2$, we can find an open set $c(g_2)$ containing them whose image by $h$ is included in $z(f)$.

We next prove that $c(g_2)$ has at most $M$ points, and this will imply that $L \leq M$, and that each point in $D_2$ is isolated. Suppose on the contrary that $c(g_2)$ has more than $M$ points. Then we can select pairwise disjoint (nonempty) clopen subsets $V_1, \ldots, V_{M+1}$ of $c(g_2)$, and $\alpha_1, \ldots, \alpha_{M+1} \in \mathbb{K}$ in such a way that there exists $g \in \mathfrak{A}(X)$ such that $Tg = \sum_{i=1}^{M+1} \alpha_i \xi_{V_i}$.

In the same way as it is proved in Proposition 3.5, we can prove here that $f \equiv 0$ outside $z(g)$. Consequently, we have that $c(g) \subset z(f)$, that is, $fg \equiv 0$, so we must have $(Tf)(Tg) \equiv 0$, which is not the case.

We conclude that there are at most $M$ different points in $c(g_2)$, and that they are isolated. ■

The following result gives us a representation of images of functions at points of $D_3$. Its proof is easy.

**Proposition 3.7.** *Let $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ or $(C(X), C(Y))$. Then $(Tf)(y) = (T\xi_X)(y) f(h(y))$ for every $f \in \mathfrak{A}(X)$ and every $y \in D_3$.*

**Proposition 3.8.** *Let $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ or $(C(X), C(Y))$. Then the set $Y \setminus D$ consists of a finite set of isolated points.*

*Proof.* First, we have that, since by Proposition 3.7, $Tf = T\xi_X \cdot f \circ h$ in $D_3$ for every $f \in \mathfrak{A}(X)$, then $(T\xi_X)(y) \neq 0$ for every $y \in D_3$.

Suppose that $D$ is not clopen. Take $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. Let $M := \operatorname{codim} R(T)$, and, for $i = 1, \ldots, M+1$, we define $\alpha_{i,1} := \alpha^i \in \mathbb{K}$. Next, also for $n \in \mathbb{N}$, we put $\alpha_{i,n} := \alpha_i^n$, and define the set

$$A_n := \left\{ y \in Y \setminus (D_1 \cup D_2) : \left|\alpha_{M+1,n}^n\right| |(T\xi_X)(y)| < 1 \right\}.$$

Obviously, by Propositions 3.5 and 3.6, each $A_n$ is a clopen subset of $Y$ containing $Y \setminus D$. Also, since $Y \setminus D$ is not clopen, then $D_3 \cap A_n$ is nonempty for every $n \in \mathbb{N}$. Now we put $B_n := A_n \setminus A_{n+1}$ for each $n \in \mathbb{N}$. Let $g_i := \sum_{n=1}^{\infty} \alpha_{i,n} \xi_{B_n} T\xi_X \in \mathfrak{A}(Y)$. It is clear that there is a (nonzero) linear combination $g := \sum_{i=1}^{M+1} \gamma_i g_i$ which belongs to $R(T)$.

This means that, if $\delta_n := \gamma_1 \alpha_{1,n} + \cdots + \gamma_{M+1} \alpha_{M+1,n} \in \mathbb{K}$ for each $n \in \mathbb{N}$, then there exists $f \in C(X)$ such that,

$$Tf \equiv \delta_n T\xi_X,$$

that is, $f \circ h = \delta_n$ on $B_n$. We also know by Proposition 1.1 (taking into account Propositions 3.5 and 3.6), that if $A$ is given as in Proposition 3.2, then

$$X \setminus A = \operatorname{cl}_X h(D \setminus D_1) = h(D_2) \cup \operatorname{cl}_X \bigcup_{n=1}^{\infty} h(B_n).$$

Now it is easy to see that $f \equiv \delta_n$ on $h(B_n)$ and $\lim_{n\to\infty} |\delta_n| = +\infty$. This implies in particular that $f$ is not bounded, so we arrive at a contradiction in the case when $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$. Consequently in this case $D$ is clopen.

When $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C(X), C(Y))$, the same implies that there exists $n_0 \in \mathbb{N}$ such that $\mathrm{cl}_X\, h(B_n) \cap \mathrm{cl}_X \bigcup_{k\neq n} h(B_k) = \emptyset$ for each $n \geq n_0$. This gives in particular that $h(B_n)$ has an open closure in $X$ for $n \geq n_0$. Also, in the same way we can see that every point in $X \setminus (A \cup h(D_2))$ belongs to the closure of one $h(B_n)$, that is, $X \setminus A = h(D_2) \cup \bigcup_{n=1}^{\infty} \mathrm{cl}_X\, h(B_n)$. Now we define, for $n \geq n_0$, a map $g' : X \to \mathbb{K}$ as

$$g' := \sum_{n=n_0}^{\infty} \alpha_{M+1,n+1}^{n+1} \xi_{\mathrm{cl}_X h(B_n)}.$$

It is easy to check that $g'$ belongs to $C(X)$, and that $|(Tg')(y)| \geq 1$ for every $y \in D_3 \cap A_{n_0}$, so $D$ is clopen.

Consequently we see that in every case, $D$ is clopen. Now we easily conclude that $Y \setminus D$ is finite, and all its points must be isolated. ∎

**Proposition 3.9.** *Let* $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$. *Then*

$$\inf\{|(T\xi_X)(y)| : y \in D_3\} > 0.$$

*Proof.* Notice that a closer look at the proof of Proposition 3.8 reveals that the contradiction we obtain for the case when $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ comes directly from the fact that no set $D_3 \cap A_n$ is empty (with no need for assuming that $D$ is not clopen). We deduce then that there exists $n_0 \in \mathbb{N}$ with $D_3 \cap A_{n_0} = \emptyset$, and we are done. ∎

**Proposition 3.10.** *Let* $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$ *or* $(C(X), C(Y))$. *Suppose that* $T$ *is injective, and that* $D_3 = Y$. *Then* $h$ *is a surjective closed map.*

*Proof.* Using Proposition 3.9 when $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C^*(X), C^*(Y))$, and the fact that $(T\xi_X)(y) \neq 0$ for every $y \in D_3 = Y$ when $(\mathfrak{A}(X), \mathfrak{A}(Y)) = (C(X), C(Y))$, it is easy to see that, taking into account the representation given in Proposition 3.7, we can assume without loss of generality that $T\xi_X = \xi_Y$.

By Proposition 1.1, the result is obvious if $Y$ is compact.

We assume that $Y$ is not compact. Again by Proposition 1.1, it is clear that the continuous map $h : Y \to X$ can be extended to a continuous surjection (which we also call $h$) $h : \beta_0 Y \to \beta_0 X$.

Let us see first that there is no point $y \in \beta_0 Y \setminus Y$ with $h(y) \in X$. Suppose on the contrary that $y_0 \in \beta_0 Y \setminus Y$ and $x_0 := h(y_0)$ belongs to $X$. Then there exists a sequence of clopen sets $(U_n)$ in $\beta_0 Y$ with $U_{n+1} \subset U_n$ and $y_0 \in U_n$ for every $n \in \mathbb{N}$, such that $Y \cap (\bigcap_{n=1}^{\infty} U_n) = \emptyset$. Let $V_n := Y \cap (U_n \setminus U_{n+1})$, for each $n \in \mathbb{N}$.

Let $M := \mathrm{codim}\, R(T) \in \mathbb{N} \cup \{0\}$. Define $\mathbb{M}_0 := \{n(M+1) : n \in \mathbb{N}\}$, $\mathbb{M}_1 := \{n(M+1)+1 : n \in \mathbb{N}\}$, ..., $\mathbb{M}_M := \{n(M+1)+M : n \in \mathbb{N}\}$. Next let $(\alpha_n)$ be a sequence in $\mathbb{K}$ (which we take in $B_{\mathbb{K}}(0,1)$ if we assume that $(\mathfrak{A}(X), \mathfrak{A}(Y)) =$

$(C^*(X), C^*(Y)))$ such that $|\alpha_n - \alpha_m| \geq 1/2$ when $n \neq m$. Set

$$\begin{aligned} g_0 &:= \sum_{n \in \mathbb{M}_0} \alpha_n \xi_{V_n} \\ g_1 &:= \sum_{n \in \mathbb{M}_1} \alpha_n \xi_{V_n} \\ &\vdots \\ g_M &:= \sum_{n \in \mathbb{M}_M} \alpha_n \xi_{V_n}. \end{aligned}$$

It is easy to see that each $g_i$ belongs to $\mathfrak{A}(Y)$. By hypothesis there are $\beta_0, \beta_1, \ldots, \beta_M \in \mathbb{K}$ (not all of them equal to 0), and $f \in \mathfrak{A}(X)$ such that $Tf = \sum_{i=0}^{M} \beta_i g_i$. Let us see that this is impossible by checking the value of $f$ at $x_0$. We are assuming that, for every $y \in Y$, $(Tf)(y) = f(h(y))$, so $f(h(y)) = \sum_{i=0}^{M} \beta_i g_i$. Let $\alpha := f(x_0)$ and fix any $\epsilon > 0$. Since $f$ is continuous, there exists a clopen neighborhood $U(x_0)$ of $x_0$ in $X$ such that $|f(x) - \alpha| < \epsilon$ for every $x \in U(x_0)$. Let $\widehat{U}$ be clopen in $\beta_0 X$ such that $U(x_0) = \widehat{U} \cap X$. Since $h : \beta_0 Y \to \beta_0 X$ is continuous, we have that $h^{-1}(\widehat{U})$ is a clopen subset of $\beta_0 Y$ which contains $y_0$. Let $V := Y \cap h^{-1}(\widehat{U})$. It is clear that, since $D_3 = Y$, if $y \in V$, then $h(y) \in U(x_0)$. This implies that for every $y \in V$, $|(Tf)(y) - \alpha| < \epsilon$. But, as in the proof of Lemma 3.4, we can see that this is not possible.

We next see that $h : D_3 \to X$ is a closed map. If $C \subset D_3$ is closed, then there exists a closed subset $C'$ of $\beta_0 D_3$ such that $C = C' \cap D_3$. Also $h(C')$ is a closed subset of $\beta_0 X$, and by the comment above, we conclude that $h(C) = h(C') \cap X$, that is, $h(C)$ is closed in $X$.

Finally, since $h(\beta_0 D_3) = \beta_0 X$, again the above remarks show that $h(D_3) = X$, and $h : D_3 \to X$ is surjective. ∎

## 4 Proof of the main results

We just prove Theorem 2.1. The proof of Theorem 2.2 is similar.

*Proof of Theorem 2.1.* For (1), see Proposition 3.2. (2) is given in Propositions 3.5, 3.6 and 3.8. Also (6) is Proposition 3.7 combined with Proposition 3.9.

Let us see now the second part of (3). Suppose that $n_1, n_2, \ldots, n_k \in \mathbb{N}$, and that we have some (pairwise disjoint) subsets of $D_3$, say $G_1 := \{y_1^1, \ldots, y_{n_1}^1\}$, ..., $G_k := \{y_1^k, \ldots, y_{n_k}^k\}$ such that $h(G_i) = x_i \in X \setminus A$, for $i \in \{1, \ldots, k\}$. Consider pairwise disjoint clopen subsets $U_i^j$ of $D_3$, such that $y_i^j \in U_i^j$ for $j = 1, \ldots, k$, and $i = 1, \ldots, n_j$. It is clear from the representation of $T$ given in (6) that no linear combination of the functions $\xi_{U_i^j}$, $j = 1, \ldots, k$, $i \geq 2$, belongs to $R(T)$. This implies in particular that just a few points $x \in X$ satisfy card $h^{-1}(\{x\}) > 1$.

Consider $X \setminus A$, where $A$ is given in (1). It is clear that the first part of (3) follows from Proposition 3.10.

Now let us prove (4). By Proposition 3.10, we know that the continuous map $h : D_3 \to X \setminus A$ is also closed and surjective. Consequently, the map $h_R : D_3/R \to$

$X \setminus A$ is a surjective homeomorphism (see for instance [4, Proposition 2.4.3 and Corollary 2.4.8]).

Let us finally prove (5). Consider the linear subspace $B := \{\xi_{D_3} Tf : f \in C^*(X)\} \subset C^*(Y)$. It is easy to see that codim $B$, the codimension of $B$ in $C^*(D_3)$, is equal to $M - \text{card}\,(Y \setminus D) - \text{card}\, D_1 - \text{card}\, D_2$, so we just need to prove that $\text{codim}\, B = \sum_{x \in X \setminus A} [\text{card}\ (D_3 \cap h^{-1}(\{x\})) - 1]$.

It is clear that $\text{codim}\, B = \text{codim}\, R(T')$, where $T' : C^*(X \setminus A) \to C^*(D_3)$ is defined, for each $f \in C^*(X \setminus A)$, as the restriction to $D_3$ of the function $Tf$. It is also easy to see that $T'$ is injective. Of course, we know that $(T'f)(y) = a(y) f(h(y))$ for every $f \in C^*(X \setminus A)$ and every $y \in D_3$, where $a = T' \xi_{X \setminus A}$. Notice that by Proposition 3.9 we can assume without loss of generality that $a \equiv 1$.

By (3), assuming $h$ defined from $D_3$ to $X \setminus A$, there are just a few points $x \in X \setminus A$ which satisfy $\text{card}\, h^{-1}(\{x\}) > 1$. We keep the notation above and suppose that these points are $x_1, \ldots, x_k$, and that $h^{-1}(\{x_i\}) = G_i \subset D_3$, for $i = 1, \ldots, k$. As above, using the clopen sets $U_i^j$, we see that $\sum_{x \in X \setminus A} [\text{card}\ (h^{-1}(\{x\})) - 1] \leq \text{codim}\, B$.

Let us finally prove the other inequality,

$$\text{codim}\, B \leq \sum_{x \in X \setminus A} \left[\text{card}\ \left(h^{-1}(\{x\})\right) - 1\right].$$

We will see that the equivalence classes of the maps $\xi_{U_i^j}$ $(i \geq 2)$ form a basis of $C^*(D_3)/R(T')$. It is easy to see that is is enough to prove that if $g \in C^*(D_3)$ satisfies to be constant on each subset $G_j$, $j = 1, \ldots, k$, then $g \in R(T')$.

Suppose then that $g \in C^*(D_3)$ satisfies $g(G_j) = \gamma_j$, for $j = 1, \ldots, k$, and let $g_R : D_3/R \to \mathbb{K}$ be such that $g = g_R \circ q$, where $q : D_3 \to D_3/R$ is the quotient map associated to $R$. We have that $g_R$ belongs to $C^*(D_3/R)$ ([4, Proposition 2.4.2]). By (4), we have that there exists $f \in C^*(X \setminus A)$ such that $g_R = f \circ h_R$. It is now easy to see that $g = f \circ h = T'f$, as we wanted to see. ∎

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