A remark on a construction of Grundhöfer

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Abstract

We consider a large class of "natural" generalizations of Grundhöfer's synthetic definition of incidence in Figueroa planes. We determine which of these generalized definitions produce projective planes and which do not. We show that those few which do produce projective planes produce only Pappian planes or Figueroa planes.

A class of non-Desarguesian, proper, finite projective planes of orders q^3 for prime powers $q \not\equiv 1 \pmod 3$ and $q > 2$ was defined by Figueroa [4] in 1982. This construction was generalized to all prime powers $q > 2$ by Hering and Schaeffer [6] later in the same year. We [2] gave a group-coset description of these finite Figueroa planes in 1983. The construction was extended to include infinite planes in 1984 by Dempwolff [3]. These constructions were all algebraic in the sense that they made essential use of collineation groups and coordinates.

In 1986 Grundhöfer [5] gave a beautiful synthetic construction which included all these Figueroa planes. It is very tempting to try to generalize this synthetic construction. However, we prove that certain kinds of generalizations of Grundhöfer's construction are impossible.

Grundhöfer's construction begins with a Pappian projective plane Π which has an order three planar automorphism α. Points and lines of Π are of three types, with respect to α, according to the structure of their orbits under $\langle \alpha \rangle$. A point P is of type-I if $P^\alpha = P$, it is of type-II if P, P^α and P^{α^2} are collinear and distinct, and it is of type-III if P, P^α and P^{α^2} are noncollinear. The types of lines are defined

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dually. Grundhöfer then defines a new incidence $\mathrm{I}_{\langle\alpha\rangle}$ (between the points and lines of Π) in terms of Pappian incidence I by:

$$P\,\mathrm{I}_{\langle\alpha\rangle}\,\ell \Longleftrightarrow \begin{cases} \ell^{\alpha}\ell^{\alpha^2}\,\mathrm{I}\,P^{\alpha}P^{\alpha^2}, & \text{if } P \text{ and } \ell \text{ are of type-III;} \\ P\,\mathrm{I}\,\ell, & \text{otherwise.} \end{cases}$$

(Note that the above definitions of point and line type and of $\mathrm{I}_{\langle\alpha\rangle}$ are unchanged if α is replaced by α^2. Thus the type of points and lines and the new incidence are uniquely defined by the order 3 planar group $\langle\alpha\rangle$. This justifies the notation $\mathrm{I}_{\langle\alpha\rangle}$.) Grundhöfer then proves synthetically that the sets of points and lines of Π together with the new incidence $\mathrm{I}_{\langle\alpha\rangle}$ is a projective plane and this plane is non-Desarguesian if the order of Π is greater than 8.

Our main result is that certain kinds of generalizations of Grundhöfer's construction do not produce new types of projective planes.

Theorem

Let $\mathbb{F}$ be a field and let α be an order h automorphism of $\mathbb{F}$. Let α also denote the automorphism of $\mathrm{PG}(2,\mathbb{F})$ *induced by componentwise action of α on homogeneous coordinates of points and lines. Let i, j, k be integers with $k \not\equiv 0 \pmod{h}$. Let $\mathcal{P}(\mathcal{L})$ denote the set of points (lines) of* $\mathrm{PG}(2,\mathbb{F})$. *Let $\mathcal{P}_{I}$ denote the set of $P \in \mathcal{P}$ which are fixed by α^k. Let $\mathcal{P}_{II}$ denote the set of $P \in \mathcal{P}$ which are not fixed by α^k and whose images under $\langle\alpha^k\rangle$ are collinear. Let $\mathcal{P}_{III}$ denote the set of $P \in \mathcal{P}$ which are not fixed by α^k and whose images under $\langle\alpha^k\rangle$ are noncollinear. Let $\mathcal{L}_{I}$, $\mathcal{L}_{II}$ and $\mathcal{L}_{III}$ be defined dually. Let $\mathrm{I}\subset \mathcal{P}\times\mathcal{L}$ denote the Pappian incidence of* $\mathrm{PG}(2,\mathbb{F})$. *Define $\mathrm{I}_{\alpha,i,j,k}$ by*

$$P\,\mathrm{I}_{\alpha,i,j,k}\,\ell \Longleftrightarrow \begin{cases} (\ell^{\alpha^j}\ell^{\alpha^{j+k}})\,\mathrm{I}\,(P^{\alpha^i}P^{\alpha^{i+k}}), & \textit{if } P\in\mathcal{P}_{III} \textit{ and } \ell\in\mathcal{L}_{III}; \\ P\,\mathrm{I}\,\ell, & \textit{otherwise.} \end{cases}$$

Then the incidence system $(\mathcal{P},\mathcal{L},\mathrm{I}_{\alpha,i,j,k})$ is a projective plane if and only if either 2 divides h and $k \equiv \frac{h}{2} \pmod{h}$ (in which case the plane is Pappian) or 3 divides h, $k \equiv \frac{h}{3}$ or $\frac{2h}{3} \pmod{h}$ and $j \equiv i \pmod{h}$ (in which case the plane is a Figueroa plane if $|\mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle| > 2$ and is a Pappian plane if $|\mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle| = 2$).

Remark: An earlier version of this theorem by the author was reported in [1] in the article by Beutlespacher on page 113. The present theorem generalizes the earlier version not only in the extended ranges of the parameters but also in the absence of special assumptions on the action of the planar automorphism on the plane.

Proof. If 2 divides h and $k \equiv \frac{h}{2} \pmod{h}$ then $\mathcal{P}_{III}$ and $\mathcal{L}_{III}$ are empty in which case the incidence system is the original Pappian plane $\mathrm{PG}(2,\mathbb{F})$.

If 3 divides h, $k \equiv \frac{h}{3}$ or $\frac{2h}{3} \pmod{h}$ and $j \equiv i \pmod{h}$ then $\mathrm{I}_{\alpha,i,j,k}$ is equivalent to $\mathrm{I}_{\alpha,0,0,k}$ which is the same as $\mathrm{I}_{\alpha^{\frac{h}{3}},0,0,1}$ which is the same as $\mathrm{I}_{\langle\alpha^{\frac{h}{3}}\rangle}$. So the claim for this case follows from Grundhöfer's result.

We prove the only if part of the theorem in three cases. But first we observe that the relation $\mathrm{I}_{\alpha,i,j,k}$ is equivalent to the relation $\mathrm{I}_{\alpha,o,j-i,k}$. So we may assume $i=0$ throughout the remainder of the proof. In two of the cases we shall use an element $t \in \mathbb{F}$ such that t has h images under $\langle\alpha\rangle$. (For example let t be an element of a normal basis of F over $\mathrm{Fix}\langle\alpha\rangle$.) To simplify notation, let $\mathrm{I}_{\alpha,0,j,k}{=}\mathrm{I}^*$.

Case 1: Assume that $k \not\equiv \dfrac{h}{2}, \dfrac{h}{3}, \dfrac{2h}{3} \pmod h$. We show in this case that there exists a pair of distinct points in $\mathcal{P}_{III}$ which are I^*-incident with two distinct lines. Let P and Q be the points with homogeneous coordinates $(1,t,t^3)$ and $(1,t,t^3+t^2)$ respectively. Also let ℓ and m be the lines with homogeneous coordinates $\begin{pmatrix} t \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -t^{\alpha^{k-j}+\alpha^{-j}+\alpha^{-k-j}} \\ t^{\alpha^{k-j}+\alpha^{-j}} + t^{\alpha^{-j}+\alpha^{-k-j}} + t^{\alpha^{k-j}+\alpha^{-k-j}} \\ 1 \end{pmatrix}$ respectively. Clearly $P \neq Q$, $\ell \neq m$, $\ell \in \mathcal{L}_{II}$ and $P\ \mathrm{I}^*\ \ell, Q\ \mathrm{I}^*\ \ell$.

It remains to show $P, Q\ \mathrm{I}^*\ m$. In order to do this we first show $P, Q \in \mathcal{P}_{III}$, $m \in \mathcal{L}_{III}$.

Clearly $P^{\alpha^k} \neq P$ by the condition on t. To prove the noncollinearity of the $\langle\alpha^k\rangle$ images of P, we first note (using $t^{\alpha^k} \neq t$) that PP^{α^k} has homogeneous coordinates $\begin{pmatrix} t^{2\alpha^k+1} + t^{\alpha^k+2} \\ -(t^{2\alpha^k} + t^{\alpha^k+1} + t^2) \\ 1 \end{pmatrix}$. Also $P^{\alpha^{2k}}\ \mathrm{I}\ PP^{\alpha^k}$ is equivalent to $0 = (t^{\alpha^{2k}} - t)(t^{\alpha^k} - t)^{\alpha^k}(t^{\alpha^{2k}} + t^{\alpha^k} + t)$, which is in turn (by the conditions on h and k) equivalent to $0 = t^{\alpha^{2k}} + t^{\alpha^k} + t$; this implies that $t^{(\alpha^k)^3} = (t^{\alpha^{2k}})^{\alpha^k} = (-t^{\alpha^k} - t)^{\alpha^k} = -t^{\alpha^{2k}} - t^{\alpha^k} = t^{\alpha^k} + t - t^{\alpha^k} = t$. From this it follows, by our choice of t, that h divides $3k$. This contradicts $k \not\equiv h, \dfrac{h}{2}, \dfrac{h}{3}, \dfrac{2h}{3} \pmod h$. Thus $P^{\alpha^{2k}}$ is not (Pappian) incident with the line PP^{α^k}. Thus the $\langle\alpha^k\rangle$ images of P are noncollinear. Thus $P \in \mathcal{P}_{III}$.

Clearly $Q^{\alpha^k} \neq Q$ by the condition on t. To prove the noncollinearity of the $\langle\alpha^k\rangle$ images of Q, we first note (using $t^{\alpha^k} \neq t$) that QQ^{α^k} has homogeneous coordinates $\begin{pmatrix} t^{2\alpha^k+1} + t^{\alpha^k+2} + t^{\alpha^k+1} \\ -(t^{2\alpha^k} + t^{\alpha^k+1} + t^2 + t^{\alpha^k} + t) \\ 1 \end{pmatrix}$. Also $Q^{\alpha^{2k}}\ \mathrm{I}\ QQ^{\alpha^k} \Leftrightarrow 0 = (t^{\alpha^{2k}} - t)(t^{\alpha^k} - t)^{\alpha^k}(t^{\alpha^{2k}} + t^{\alpha^k} + t + 1) \Leftrightarrow$ (by the conditions on h and k) $0 = t^{\alpha^{2k}} + t^{\alpha^k} + t + 1 \Rightarrow t^{(\alpha^k)^3} = (t^{\alpha^{2k}})^{\alpha^k} = (-t^{\alpha^k} - t - 1)^{\alpha^k} = -t^{\alpha^{2k}} - t^{\alpha^k} - 1 = t^{\alpha^k} + t + 1 - t^{\alpha^k} - 1 = t$. From this it follows, by our choice of t, that h divides $3k$. This contradicts $k \not\equiv h, \dfrac{h}{2}, \dfrac{h}{3}, \dfrac{2h}{3} \pmod h$. Thus $Q^{\alpha^{2k}}$ is not (Pappian) incident with the line QQ^{α^k}. Thus the $\langle\alpha^k\rangle$ images of Q are noncollinear. Thus $Q \in \mathcal{P}_{III}$.

If $m^{\alpha^k} = m$, then (using the first coordinate of m) $t^{\alpha^{2k}} = t^{\alpha^{-k}}$. This implies $t^{(\alpha^k)^3} = t$ which is a contradiction as before. To prove the nonconfluence of the $\langle\alpha^k\rangle$ images of m, we first note (using $t^{\alpha^{2k}} \neq t^{\alpha^{-k}}$) that mm^{α^k} has homogeneous coordinates $(t^{\alpha^{k-j}} + t^{\alpha^{-j}}, t^{\alpha^{k-j}+\alpha^{-j}}, -t^{2\alpha^{k-j}+2\alpha^{-j}})$. From this it follows that $mm^{\alpha^k}\ \mathrm{I}\ m^{\alpha^{2k}} \Leftrightarrow 0 = t^{2\alpha^{k-j}}(t^{\alpha^{3k-j}} - t^{\alpha^{-j}})(-t^{\alpha^{2k-j}} + t^{\alpha^{-j}})$ which can never happen under our conditions on k and h. Thus the $\langle\alpha^k\rangle$ images of m are nonconfluent. Thus $m \in \mathcal{L}_{III}$.

From our coordinates of PP^{α^k}, QQ^{α^k} and mm^{α^k} and from the definition of I^* it now follows that P, Q are I^*-incident with m.

Case 2: Assume 3 divides h, $k \equiv \frac{h}{3}$ or $\frac{2h}{3} \pmod h$, $j \not\equiv 0 \pmod h$ and $|\mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle| > 2$. (The previous argument will not work here because, as we shall see below, in this case any two points from $\mathcal{P}_{III}$ are I^*-incident with a unique line.)

It is easy to verify that the relations $\mathrm{I}_{\alpha,0,j,k}$, $\mathrm{I}_{\alpha,0,j,2k}$, and $\mathrm{I}_{\alpha,0,j,\frac{h}{3}}$ are mutually equivalent. So it is sufficient to assume $k = \frac{h}{3}$.

We now observe that our definitions allow us to relate our I^* incidence and Grundhöfer's Figueroa incidence as follows:

$$P \,\mathrm{I}^*\, \ell \iff \begin{cases} P \,\mathrm{I}_{\langle\alpha^{\frac{h}{3}}\rangle}\, \ell^{\alpha^j}, & \text{if } P \in \mathcal{P}_{III} \text{ and } \ell \in \mathcal{L}_{III}; \\ P \,\mathrm{I}_{\langle\alpha^{\frac{h}{3}}\rangle}\, \ell, & \text{otherwise.} \end{cases}$$

Using the fact that $\mathrm{I}_{\langle\alpha^{\frac{h}{3}}\rangle}$ defines a projective plane, we see that there is a unique line I^*-incident with any two distinct points except possibly in the case of one point from $\mathcal{P}_{III}$ and the other from $\mathcal{P}_{II}$. This case will provide our contradiction.

A slightly different description of I^* is useful in deducing this contradiction

$$P \,\mathrm{I}^*\, \ell \iff \begin{cases} P^{\alpha^{-j}} \,\mathrm{I}_{\langle\alpha^{\frac{h}{3}}\rangle}\, \ell, & \text{if } P \in \mathcal{P}_{III} \text{ and } \ell \in \mathcal{L}_{III}; \\ P \,\mathrm{I}\, \ell, & \text{otherwise.} \end{cases}$$

Let $P \in \mathcal{P}_{III}$ and $Q \in \mathcal{P}_{II}$. Clearly P and Q are never I^*-incident with a line of $\mathcal{L}_I$ and they are I^*-incident with a line of $\mathcal{L}_{II}$ if and only if they are I-incident with that line. Finally P and Q are I^*-incident with a line of $\mathcal{L}_{III}$ if and only if $P^{\alpha^{-j}}$ and Q are $\mathrm{I}_{\langle\alpha^{\frac{h}{3}}\rangle}$-incident with that line. Because $\mathrm{I}_{\langle\alpha^{\frac{h}{3}}\rangle}$ defines a projective plane, this occurs if and only if $P^{\alpha^{-j}}$ and Q are not $\mathrm{I}_{\langle\alpha^{\frac{h}{3}}\rangle}$-incident (i.e. not I-incident) with a line of $\mathcal{L}_{II}$. We shall show that I^* does not define a plane by proving that there exist points $P \in \mathcal{P}_{III}$ and $Q \in \mathcal{P}_{II}$ such that P and Q are I-incident with a line of $\mathcal{L}_{II}$ and that $P^{\alpha^{-j}}$ and Q are not I-incident with a line of $\mathcal{L}_{II}$.

Let $x \in \mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle$. Let $a \in \mathbb{F}$ be such that the element $a(t - t^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{h}{3}}}$ has nonzero relative trace with respect to $\mathbb{F}$ over $\mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle$. Note that this implies that $a^{\alpha^{\frac{h}{3}}} \neq a$. Let P_x and Q be points with homogeneous coordinates $(a + x, -1, t)$ and $(0, -1, t)$ respectively. Also let ℓ and m_x be the lines with homogeneous coordinates $\begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix}$ and $\begin{pmatrix} t - t^{\alpha^{-j}} \\ t(a^{\alpha^{-j}} + x^{\alpha^{-j}}) \\ a^{\alpha^{-j}} + x^{\alpha^{-j}} \end{pmatrix}$ respectively. Our conditions imply $P_x \in \mathcal{P}_{III}$, $Q \in \mathcal{P}_{II}$, $\ell \in \mathcal{L}_{II}$, $P_x \,\mathrm{I}\, \ell, Q \,\mathrm{I}\, \ell$, $Q \,\mathrm{I}\, m_x$ and $P_x^{\alpha^{-j}} \,\mathrm{I}\, m_x$.

It remains to show that for some choice of x it is true that $m_x \in \mathcal{L}_{III}$. Suppose otherwise i.e. suppose $m_x, m_x^{\alpha^{\frac{h}{3}}}, m_x^{\alpha^{\frac{2h}{3}}}$ are confluent for all allowed values of x. This is equivalent to a quadratic polynomial in x with coefficients in $\mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle$ being zero for all allowed values of x. Letting Tr denote the relative trace function, $b = a^{\alpha^{\frac{2h}{3}}} + a^{\alpha^{\frac{h}{3}}}$,

$c = a^{\alpha^{\frac{2h}{3}}+\alpha^{\frac{h}{3}}}$, $s = (t^{\alpha^j} - t)(t - t^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{h}{3}+j}}$, this polynomial becomes

$$x^2 \operatorname{Tr}(s) + x \operatorname{Tr}(sb) + \operatorname{Tr}(sc)$$

For $|\mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle| > 2$ the only way that this polynomial can vanish for all $x \in \mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle$ is for all of its coefficients to be zero. However the coefficients all vanish $\Leftrightarrow s^{\alpha^{\frac{h}{3}}} = s(b - b^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{2h}{3}} - \alpha^{\frac{h}{3}}}$ and $0 = s\Big((b - b^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{h}{3}}}(c - c^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{2h}{3}}} - (b - b^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{2h}{3}}}(c - c^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{h}{3}}}\Big) \Leftrightarrow s^{\alpha^{\frac{h}{3}}} = s(a - a^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{2h}{3}} - \alpha^{\frac{h}{3}}}$ and $0 = s\Big((a - a^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{h}{3}}} a^{\alpha^{\frac{h}{3}}}(a - a^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{2h}{3}}} - (a - a^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{2h}{3}}} a(a - a^{\alpha^{\frac{h}{3}}})^{\alpha^{\frac{h}{3}}}\Big)$. The last of these equations simplifies to $0 = s(a - a^{\alpha^{\frac{h}{3}}})^{1+\alpha^{\frac{h}{3}}+\alpha^{\frac{2h}{3}}}$. This implies $s = 0$ which contradicts our choice of t and the fact that $j \not\equiv 0 \pmod h$.

Case 3: Assume 3 divides h, $k \equiv \frac{h}{3}$ or $\frac{2h}{3} \pmod h$, $j \not\equiv 0 \pmod h$ and $|\mathrm{Fix}\langle\alpha^{\frac{h}{3}}\rangle| = 2$. Then $|\mathbb{F}| = 2^3$ and $h = 3$. By the definitions of the subsets of points and lines and by counting, each point P (line ℓ) of $\mathcal{P}_{III}$ ($\mathcal{L}_{III}$) is I-incident with exactly zero lines (points) of $\mathcal{L}_I$ ($\mathcal{P}_I$) and exactly seven lines (points) of $\mathcal{L}_{II}$ ($\mathcal{P}_{II}$). These seven lines (points) are the lines YP (points $y\ell$) with $Y \mathrel{\mathrm{I}} P^\alpha P^{\alpha^2}$, $Y \neq P^\alpha, P^{\alpha^2}$ ($\ell^\alpha \ell^{\alpha^2} \mathrel{\mathrm{I}} y$, $y \neq \ell^\alpha, \ell^{\alpha^2}$). By the definition of I^*, $P \mathrel{\mathrm{I}^*} P^\alpha P^{\alpha^2}$. But for any of the above described Y, $Y \mathrel{\mathrm{I}^*} P^\alpha P^{\alpha^2}$ and $Y \mathrel{\mathrm{I}^*} YP$. Also $P \mathrel{\mathrm{I}^*} YP$ because $YP \in \mathcal{L}_{II}$. So P and Y are two distinct points which are both I^*-incident with the two distinct lines $P^\alpha P^{\alpha^2}$ and YP. ■

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