The Cone Length and Category of Maps: Pushouts, Products and Fibrations

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Abstract

For any collection of spaces $\mathcal{A}$, we investigate two non-negative integer homotopy invariants of maps: $L_{\mathcal{A}}(f)$, the $\mathcal{A}$-cone length of f, and $\mathcal{L}_{\mathcal{A}}(f)$, the $\mathcal{A}$-category of f. When $\mathcal{A}$ is the collection of all spaces, these are the cone length and category of f, respectively, both of which have been studied previously. The following results are obtained: (1) For a map of one homotopy pushout diagram into another, we derive an upper bound for $L_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ of the induced map of homotopy pushouts in terms of $L_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ of the other maps. This has many applications, including an inequality for $L_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ of the maps in a mapping of one mapping cone sequence into another. (2) We establish an upper bound for $L_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ of the product of two maps in terms of $L_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ of the given maps and the $\mathcal{A}$-cone length of their domains. (3) We study our invariants in a pullback square and obtain as a consequence an upper bound for the $\mathcal{A}$-cone length and $\mathcal{A}$-category of the total space of a fibration in terms of the $\mathcal{A}$-cone length and $\mathcal{A}$-category of the base and fiber. We conclude with several remarks, examples and open questions.

1 Introduction

In this paper we continue our investigation, begun in [A-S-S], of the cone length and category of maps relative to a fixed collection of spaces. For a collection $\mathcal{A}$ of spaces we consider two non-negative integer homotopy invariants of maps: the $\mathcal{A}$-category,

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denoted $\mathcal{L}_\mathcal{A}$, and the $\mathcal{A}$-cone length, denoted $L_\mathcal{A}$. When $\mathcal{A}$ is the collection of all spaces, $\mathcal{L}_\mathcal{A}(f)$ is the category of the map f as defined and studied in [Fa-Hu] and [Co2] and $L_\mathcal{A}(f)$ is the cone length of the map f as defined and studied in [Mar2] and [Co1]. If, in this special case, f is the inclusion of the base point into Y, then $\mathcal{L}_\mathcal{A}(f)$ is just the category of the space Y, which was introduced in 1934 by Lusternik and Schnirelmann in their work on the number of critical points of smooth functions on a manifold [L-S]. In addition, $L_\mathcal{A}(f)$ is the cone length of Y which has been studied by several people [Co1, Co3, Co4, Ga1, Mar2, St1, Ta] in the context of homotopy theory. For an arbitrary collection $\mathcal{A}$, the $\mathcal{A}$-category and the $\mathcal{A}$-cone length of the inclusion of the basepoint into Y coincide with the $\mathcal{A}$-category and $\mathcal{A}$-cone length of Y. Variants of this concept have been studied previously [S-T].

Thus our invariants are common generalizations of the category and cone length of a map and the $\mathcal{A}$-category and $\mathcal{A}$-cone length of a space. In addition to providing a general framework for many existing notions and retrieving known results as special cases, they have led to several new concepts and results. To discuss this, we first briefly summarize that part of our previous work which is relevant to this paper. More details are given in §2.

In [A-S-S] we introduced, for a fixed collection $\mathcal{A}$, five simple axioms which an integer valued function of based maps may satisfy. Then $\mathcal{L}_\mathcal{A}$ was defined as the maximum of all such functions. Similarly, $L_\mathcal{A}$ was defined as the maximum of all functions which satisfy an analogous set of five axioms. We then gave alternate characterizations of these invariants in terms of certain decompositions of maps. For instance, $L_\mathcal{A}(f)$ is essentially the smallest integer n such that f admits a decomposition up to homotopy as

$$X = X_0 \xrightarrow{j_0} X_1 \xrightarrow{j_1} \cdots \xrightarrow{j_{n-1}} X_n = Y,$$

where $L_i \longrightarrow X_i \xrightarrow{j_i} X_{i+1}$ is a mapping cone sequence with $L_i \in \mathcal{A}$. When f is the inclusion of the base point into Y, we obtain $\mathrm{cat}_\mathcal{A}(Y)$ and $\mathrm{cl}_\mathcal{A}(Y)$, as noted above, and when g is the map of X to a one point space, we obtain two new invariants of spaces, the $\mathcal{A}$-kitegory of X, $\mathrm{kit}_\mathcal{A}(X) = \mathcal{L}_\mathcal{A}(g)$, and the $\mathcal{A}$-killing length of X, $\mathrm{kl}_\mathcal{A}(X) = L_\mathcal{A}(g)$. In [A-S-S] we made a preliminary study of these invariants and their interrelations.

From the time the concept of category was first introduced to the present, many people have been interested in the following questions: What is the relationship between the categories of the spaces which appear in a homotopy pushout [Mar2, Ha1]? What is the category of the product of two spaces in terms of the categories of the factors [A-Sta, Bas, C-P2, Ga2, Iw, Ro, St2, Ta, Van]? What is the relationship of the categories of the spaces which appear in a fiber sequence [Ha2, J-S, Var]? Similar questions have also been considered for cone length. In this paper we study these questions for the $\mathcal{A}$-category and $\mathcal{A}$-cone length of maps, and provide reasonably complete answers. This gives both new and known results for the $\mathcal{A}$-category and $\mathcal{A}$-cone length of spaces as well as new results for the $\mathcal{A}$-kitegory and the $\mathcal{A}$-killing length.

We now summarize the contents of the paper. In §2 we give our terminology and notation and discuss our earlier work in more detail. In §3 we prove one of our main results, the Homotopy Pushout Mapping Theorem. This theorem gives an inequality for the $\mathcal{A}$-categories and the $\mathcal{A}$-cone lengths of the four maps which constitute a map of one homotopy pushout square into another. Many applications are given in §4. In particular, we obtain results about the $\mathcal{A}$-categories and $\mathcal{A}$-cone lengths of the maps which appear in a mapping of one mapping cone sequence into another. In §5 we establish an upper bound for the $\mathcal{A}$-category (resp., $\mathcal{A}$-cone length) of the product of two maps in terms of the $\mathcal{A}$-categories (resp., $\mathcal{A}$-cone lengths) of the original maps and the $\mathcal{A}$-cone lengths of their domains. By specializing to spaces and letting $\mathcal{A}$ be the collection of all spaces, we retrieve classical results on the category and cone length of the product of two spaces. We study pullbacks in §6. As a consequence of our main result on pullbacks we obtain an inequality for the $\mathcal{A}$-category of the total space of a fiber sequence in terms of the $\mathcal{A}$-category of the base and fiber, and a similar result for $\mathcal{A}$-cone length. Section 7 contains a potpourri of results, examples and questions. We begin by presenting a few simple, but useful, results about $\mathcal{A}$-category and $\mathcal{A}$-cone length. We then give some examples to illustrate the difference between these invariants for different collections $\mathcal{A}$. In particular, we show that some results that are known for the collection of all spaces do not hold for arbitrary collections. Finally, we state and discuss some open problems.

We conclude this section by emphasizing two important points. First of all, there are several different notions of the category of a map in the literature. The one that we generalize here to the $\mathcal{A}$-category of a map has been studied in [Fa-Hu] and [Co2]. It is not the same as the one considered in [Fo, B-G]. In addition, Clapp and Puppe have considered the category of a map with respect to a collection of spaces [C-P1, C-P2]. However, their notion is completely different from ours. Secondly, although we state and prove our results in the category of well-pointed spaces and based maps, it should be clear that nearly all our results hold in a (closed) model category [Qu] and that all of our results hold in a J-category [Do1, H-L].

2 Preliminaries

In this section we give our notation and terminology and also recall some results from [A-S-S] which will be needed later.

All topological spaces are based and have the based homotopy type of CW-complexes, though we could more generally consider well-pointed based spaces. All maps and homotopies are to preserve base points. We *do* distinguish between a map and a homotopy class. By a commutative diagram we mean one which is *strictly* commutative.

We next give some notation which is standard for homotopy theory: $*$ denotes the base point of a space or the space consisting of a single point, $\simeq$ denotes homotopy of maps and $\equiv$ denotes same homotopy type of spaces. We let $0 : X \longrightarrow Y$ stand for the constant map and id: $X \longrightarrow X$ for the identity map. We use Σ for (reduced) suspension, $*$ for (reduced) join, $\vee$ for wedge sum and $\wedge$ for smash product.

We call a sequence $A \xrightarrow{f} X \xrightarrow{j} C$ of spaces and maps a *mapping cone sequence* if C is the mapping cone of f and j is the standard inclusion. Then j is a cofibration with cofiber ΣA. Using the mapping cylinder construction, we see that the concept of a cofiber sequence and the concept of a mapping cone sequence are equivalent

[Hi, Ch. 3]. For maps $C \xleftarrow{g} A \xrightarrow{f} B$ we can form the homotopy pushout Q

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & Q \end{tikzcd}$$

by defining Q to be the quotient of $B \vee (A \times I) \vee C$ under the equivalence relation $(*, t) \sim *$, $(a, 0) \sim f(a)$ and $(a, 1) \sim g(a)$ for $t \in I$ and $a \in A$. Note that $A \xrightarrow{f} X \xrightarrow{j} C$ is a mapping cone sequence if and only if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d, "j"] \\ * \arrow[r] & C \end{tikzcd}$$

is a homotopy pushout square. The pullback P of $C \xrightarrow{g} A \xleftarrow{f} B$ is defined by

$$P = \{(b, c) \,|\, b \in B, c \in C, f(b) = g(c)\} \subseteq B \times C.$$

We only use this construction when f is a fibration. Thus all our pullbacks are homotopy pullbacks as well. Given a map $f : X \longrightarrow Y$ we say that a map $g : X' \longrightarrow Y'$ *homotopy dominates* f (or f is a *homotopy retract* of g) if there is a homotopy-commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "j"] & Y' \arrow[r, "s"] & Y \end{tikzcd}$$

such that $ri \simeq \mathrm{id}$ and $sj \simeq \mathrm{id}$. If the diagram is strictly commutative and both homotopies are equality, we delete the word 'homotopy' from the definition. If g homotopy dominates f as above and in addition $ir \simeq \mathrm{id}$ and $js \simeq \mathrm{id}$ (i.e., r and s are homotopy equivalences with homotopy inverses i and j), we say that f and g are *homotopy equivalent.*

Next we recall some definitions and results from [A-S-S] which will be used in the sequel. By a *collection* $\mathcal{A}$ we mean a class of spaces containing $*$ such that if $A \in \mathcal{A}$ and $A \equiv A'$, then $A' \in \mathcal{A}$. We say that (1) $\mathcal{A}$ is *closed under suspension* if $A \in \mathcal{A}$ implies $\Sigma A \in \mathcal{A}$, (2) $\mathcal{A}$ is *closed under wedges* if $A, A' \in \mathcal{A}$ implies $A \vee A' \in \mathcal{A}$ and (3) $\mathcal{A}$ is *closed under joins* if $A, A' \in \mathcal{A}$ implies $A * A' \in \mathcal{A}$. Examples of collections that we consider are (1) the collection $\mathcal{A} = \{\text{all spaces}\}$ of all spaces, (2) the collection Σ of all suspensions, and (3) the collection $\mathcal{S}$ of all wedges of spheres (including S^0).

Let $\mathcal{A}$ be a collection and $\ell_{\mathcal{A}}$ a function which assigns to each map f an integer $0 \leq \ell_{\mathcal{A}}(f) \leq \infty$. We say that $\ell_{\mathcal{A}}$ satisfies the $\mathcal{A}$-*cone axioms* if

(1) (Homotopy Axiom) If $f \simeq g$, then $\ell_{\mathcal{A}}(f) = \ell_{\mathcal{A}}(g)$.

(2) (Normalization Axiom) If f is a homotopy equivalence, then $\ell_{\mathcal{A}}(f) = 0$.

(3) (Composition Axiom) $\ell_{\mathcal{A}}(fg) \leq \ell_{\mathcal{A}}(f) + \ell_{\mathcal{A}}(g)$.

(4) (Mapping Cone Axiom) If $A \longrightarrow X \xrightarrow{f} Y$ is a mapping cone sequence with $A \in \mathcal{A}$, then $\ell_{\mathcal{A}}(f) \leq 1$.

(5) (Equivalence Axiom) If f and g are homotopy equivalent, then $\ell_{\mathcal{A}}(f) = \ell_{\mathcal{A}}(g)$.

We say that $\ell_{\mathcal{A}}$ satisfies the $\mathcal{A}$-*category axioms* if $\ell_{\mathcal{A}}$ satisfies (1) – (4) and

(5′) (Domination Axiom) If f is dominated by g, then $\ell_{\mathcal{A}}(f) \leq \ell_{\mathcal{A}}(g)$.

Definition 2.1. We denote by $L_{\mathcal{A}}(f)$ the maximum of all $\ell_{\mathcal{A}}(f)$ where $\ell_{\mathcal{A}}$ satisfies (1)–(5) and by $\mathcal{L}_{\mathcal{A}}(f)$ the maximum of all $\ell_{\mathcal{A}}(f)$ where $\ell_{\mathcal{A}}$ satisfies (1)–(4) and (5′). We call $L_{\mathcal{A}}(f)$ the $\mathcal{A}$-*cone length* of f and $\mathcal{L}_{\mathcal{A}}(f)$ the $\mathcal{A}$-*category* of f.

Since (5) is weaker than (5′), $\mathcal{L}_{\mathcal{A}}(f) \leq L_{\mathcal{A}}(f)$. In [A-S-S] it is proved that when $\mathcal{A} = \{\text{all spaces}\}$, $L_{\mathcal{A}}(f)$ is the cone length of f as defined in [Co2, Mar2], and $\mathcal{L}_{\mathcal{A}}(f)$ is the category of f as defined in [Fa-Hu, Co2].

One of the main results of [A-S-S] gives alternate characterizations of $L_{\mathcal{A}}(f)$ and $\mathcal{L}_{\mathcal{A}}(f)$ in terms of decompositions of the map f. If $f : X \longrightarrow Y$ is a map, then an $\mathcal{A}$-*cone decomposition of length* n of f is a homotopy-commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "j_0"] \arrow[d, equal] & X_1 \arrow[r, "j_1"] & \cdots \arrow[r, "j_{n-2}"] & X_{n-1} \arrow[r, "j_{n-1}"] & X_n \arrow[d, shift right, "f_n"'] \\
X \arrow[rrrr, "f"] & & & & Y \arrow[u, shift right, "s"']
\end{tikzcd}$$

in which f_n is a homotopy equivalence with homotopy inverse s and each map j_i is part of a mapping cone sequence $A_i \longrightarrow X_i \xrightarrow{j_i} X_{i+1}$ with $A_i \in \mathcal{A}$. Thus $f_n j_{n-1} \cdots j_0 \simeq f$, $sf \simeq j_{n-1} \cdots j_0$, $f_n s \simeq \mathrm{id}$ and $sf_n \simeq \mathrm{id}$. The homotopy-commutative diagram above is an $\mathcal{A}$-*category decomposition of* f *of length* n if s is simply a homotopy section of f_n, i.e., if $f_n j_{n-1} \cdots j_0 \simeq f$, $sf \simeq j_{n-1} \cdots j_0$ and $f_n s \simeq \mathrm{id}$, but sf_n need not be homotopic to the identity. We prove in [A-S-S, Thm. 3.7] that

$$L_{\mathcal{A}}(f) = \begin{cases} 0 & \text{if } f \text{ is a homotopy equivalence} \\ \infty & \text{if there is no } \mathcal{A}\text{−cone decomposition of } f \\ n & \text{if } n \text{ is the smallest integer such that there exists an} \\ & \mathcal{A}\text{−cone decomposition of length } n \text{ of } f. \end{cases}$$

We similarly characterize $\mathcal{L}_{\mathcal{A}}(f)$ using $\mathcal{A}$-category decompositions instead of $\mathcal{A}$-cone decompositions. Observe that if the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ on path components is not surjective, then $L_{\mathcal{A}}(f)$ and $\mathcal{L}_{\mathcal{A}}(f)$ are infinite for *every* collection $\mathcal{A}$.

We have also studied four numerical invariants of spaces, defined in terms of the invariants $L_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$ as follows:

$$\begin{array}{rclcrcl} \mathrm{cl}_{\mathcal{A}}(X) & = & L_{\mathcal{A}}(* \longrightarrow X) & \qquad & \mathrm{cat}_{\mathcal{A}}(X) & = & \mathcal{L}_{\mathcal{A}}(* \longrightarrow X) \\ \mathrm{kl}_{\mathcal{A}}(X) & = & L_{\mathcal{A}}(X \longrightarrow *) & & \mathrm{kit}_{\mathcal{A}}(X) & = & \mathcal{L}_{\mathcal{A}}(X \longrightarrow *). \end{array}$$

When $\mathcal{A} = \{\text{all spaces}\}$, $\mathrm{cat}_{\mathcal{A}}(X) = \mathrm{cat}(X)$, the reduced Lusternik-Schnirelmann category of X [A-S-S, Prop. 4.1], and $\mathrm{cl}_{\mathcal{A}}(X) = \mathrm{cl}(X)$, the cone length of X. Moreover, $\mathrm{kit}_{\mathcal{A}}(X) \leq 1$ and $\mathrm{kl}_{\mathcal{A}}(X) \leq 1$ for every space X in this case.

3 The Homotopy Pushout Mapping Theorem

In this section we prove the first main result of this paper. This consists of two inequalities, one for the $\mathcal{A}$-cone length and one for the $\mathcal{A}$-category of the maps from one homotopy pushout square to another. In Section 4 we will derive numerous consequences of this result.

We begin with a technical result that plays a key role in the proof of the main theorem.

Lemma 3.1. *Let $f : X \longrightarrow Y$ be a map with $\mathcal{L}_{\mathcal{A}}(f) = n$. Then there exists a map $g : X \longrightarrow Z$ and maps $i : Y \longrightarrow Z$ and $r : Z \longrightarrow Y$ such that the diagram*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "f"] & \\ Y \arrow[r, "i"'] & Z \arrow[r, "r"'] & Y, \end{tikzcd}$$

commutes, $ri = \mathrm{id}$ and $L_{\mathcal{A}}(g) = n$.

Proof By [A-S-S, Cor. 4.4] there is a homotopy-commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "f"] & \\ Y \arrow[r, "j"'] & W \arrow[r, "s"'] & Y \end{tikzcd}$$

with $sj \simeq \mathrm{id}$ and $L_{\mathcal{A}}(h) = n$. We factor s as $W \xrightarrow{s^0} E \xrightarrow{s^1} Y$, where s^0 is a homotopy equivalence and s^1 is fibration [Hi, Ch. 3]. Then $s^0 j : Y \longrightarrow E$ and $s^1 s^0 j \simeq \mathrm{id}$. Thus there is a map $t : Y \longrightarrow E$ such that $t \simeq s^0 j$ and $s^1 t = \mathrm{id}$. Now we have a homotopy-commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "s^0 h"'] \arrow[dr, "f"] & \\ Y \arrow[r, "t"'] & E \arrow[r, "s^1"'] & Y. \end{tikzcd}$$

This proves the lemma with $g = tf$, $Z = E$, $i = t$ and $r = s^1$. ∎

Theorem 3.2. *Let $\mathcal{A}$ be a collection of spaces that is closed under wedges and suspension and let*

$$\begin{tikzcd} C \arrow[d, "c"] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\ C' & A' \arrow[l, "g'"] \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a commutative diagram. Let D be the homotopy pushout of the top row, D' be the homotopy pushout of the bottom row, and $d : D \longrightarrow D'$ the induced map. Then

1. $L_{\mathcal{A}}(d) \leq L_{\mathcal{A}}(a) + \max(L_{\mathcal{A}}(b), L_{\mathcal{A}}(c))$;
2. $\mathcal{L}_{\mathcal{A}}(d) \leq \mathcal{L}_{\mathcal{A}}(a) + \max(\mathcal{L}_{\mathcal{A}}(b), \mathcal{L}_{\mathcal{A}}(c))$.

Proof First we prove (1). We factor the given diagram as

$$\begin{tikzcd} C \arrow[d, "c"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, "b"] \\ C' \arrow[d, equal] & A \arrow[l, "cg"'] \arrow[r, "bf"] \arrow[d, "a"] & B' \arrow[d, equal] \\ C' & A' \arrow[l, "g'"'] \arrow[r, "f'"] & B' \end{tikzcd}$$

and let $\overline{D}$ denote the homotopy pushout of the middle row. Then we have a factorization

$$\begin{tikzcd} D \arrow[rr, "d"] \arrow[dr, "d'"'] & & D' \\ & \overline{D} \arrow[ur, "d''"'] & \end{tikzcd}$$

of d. Since $L_{\mathcal{A}}(d) \le L_{\mathcal{A}}(d') + L_{\mathcal{A}}(d'')$ by the Composition Axiom, it suffices to prove the result in the two special cases

(a) $A = A'$ and $a = \mathrm{id}$,

(b) $B = B'$, $C = C'$, $b = \mathrm{id}$ and $c = \mathrm{id}$.

We begin with (a) and let $m = \max(L_{\mathcal{A}}(b), L_{\mathcal{A}}(c))$. We consider an $\mathcal{A}$-cone decomposition of b of length m. This yields a homotopy factorization of $b \simeq hi_{m-1} \cdots i_1 i_0$:

$$B = X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{m-1}} X_m \xrightarrow{h} B',$$

where $A_l \longrightarrow X_l \xrightarrow{i_l} X_{l+1}$ is a mapping cone sequence with $A_l \in \mathcal{A}$ for each l and h is a homotopy equivalence. Since $i_{m-1} \cdots i_1 i_0$ is a cofibration, h is homotopic to a map (also called h) such that $b = hi_{m-1} \cdots i_1 i_0$. Similarly, we have an $\mathcal{A}$-cone decomposition of c of length m which gives a factorization $c = kj_{m-1} \cdots j_1 j_0$:

$$C = Y_0 \xrightarrow{j_0} Y_1 \xrightarrow{j_1} \cdots \xrightarrow{j_{m-1}} Y_m \xrightarrow{k} C',$$

where $B_l \longrightarrow Y_l \xrightarrow{j_l} Y_{l+1}$ is a mapping cone sequence with $B_l \in \mathcal{A}$ for each l and k

is a homotopy equivalence. Thus we have a commutative diagram

$$\begin{tikzcd}
Y_0 = C \arrow[d, "j_0"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, equal] & B = X_0 \arrow[d, "i_0"] \\
Y_1 \arrow[d, "j_1"'] & A \arrow[l, "j_0 g"'] \arrow[r, "i_0 f"] \arrow[d, equal] & X_1 \arrow[d, "i_1"] \\
\vdots \arrow[d] & \vdots \arrow[d, equal] & \vdots \arrow[d] \\
Y_{m-1} \arrow[d, "j_{m-1}"'] & A \arrow[l, "j_{m-2}\cdots j_0 g"'] \arrow[r, "i_{m-2}\cdots i_0 f"] \arrow[d, equal] & X_{m-1} \arrow[d, "i_{m-1}"] \\
Y_m \arrow[d, "k"'] & A \arrow[l, "j_{m-1}\cdots j_0 g"'] \arrow[r, "i_{m-1}\cdots i_0 f"] \arrow[d, equal] & X_m \arrow[d, "h"] \\
C' & A \arrow[l, "cg"'] \arrow[r, "bf"] & B'.
\end{tikzcd}$$

We number the rows $0, 1, \ldots, m+1$ and let D_l be the homotopy pushout of the l^{th} row, with induced maps $d_l : D_l \longrightarrow D_{l+1}$. Then $D_0 = D$, $D_{m+1} = \overline{D}$ and $d_m \cdots d_0 = d' : D \longrightarrow \overline{D}$. Thus it suffices to prove

(i) $L_{\mathcal{A}}(d_l) \leq 1$ for $l = 0, \ldots, m-1$,

(ii) $L_{\mathcal{A}}(d_m) = 0$.

We first establish (i). Consider the commutative diagram

$$\begin{tikzcd}
B_l \arrow[d] & * \arrow[l] \arrow[r] \arrow[d] & A_l \arrow[d] & & A_l \vee B_l \arrow[d] \\
Y_l \arrow[d, "j_l"'] & A \arrow[l] \arrow[r] \arrow[d, equal] & X_l \arrow[d, "i_l"] \arrow[rr, dotted, "\text{homotopy pushout}"] & & D_l \arrow[d] \\
Y_{l+1} & A \arrow[l] \arrow[r] & X_{l+1} & & D_{l+1},
\end{tikzcd}$$

where the columns are regarded as mapping cone sequences. The homotopy pushouts of the rows form a sequence $A_l \vee B_l \longrightarrow D_l \longrightarrow D_{l+1}$. By the Four Cofibrations Theorem, this is a cofiber sequence (see [Do2, p. 21]). Since $\mathcal{A}$ is closed under wedges, $A_l \vee B_l \in \mathcal{A}$. Therefore $L_{\mathcal{A}}(d_l) \leq 1$ by the Mapping Cone Axiom. For (ii) we note that $d_m : D_m \longrightarrow \overline{D}$ is a homotopy equivalence since h and k are homotopy equivalences [B-K, Ch. XII, § 4.2]. Thus $L_{\mathcal{A}}(d_m) = 0$, which completes the proof of (a).

For (b) we proceed similarly by assuming that $L_{\mathcal{A}}(a) = m$ and taking an $\mathcal{A}$-cone length decomposition of a of length m:

$$A = X_0 \xrightarrow{i_0} X_1 \longrightarrow \cdots \longrightarrow X_{m-1} \xrightarrow{i_{m-1}} X_m \xrightarrow{h} A',$$

where $A_l \longrightarrow X_l \xrightarrow{i_l} X_{l+1}$ is a mapping cone sequence with $A_l \in \mathcal{A}$, h a homotopy equivalence and $a = hi_{m-1} \cdots i_0$. This yields a commutative diagram

$$\begin{tikzcd}
C' \arrow[d, equal] & A = X_0 \arrow[l, "cg"'] \arrow[r, "bf"] \arrow[d, "i_0"] & B' \arrow[d, equal] \\
C' \arrow[d, equal] & X_1 \arrow[l, "g'hi_{m-1}\cdots i_1"'] \arrow[r, "f'hi_{m-1}\cdots i_1"] \arrow[d, "i_1"] & B' \arrow[d, equal] \\
\vdots \arrow[d, equal] & \vdots \arrow[d] & \vdots \arrow[d, equal] \\
C' \arrow[d, equal] & X_{m-1} \arrow[l, "g'hi_{m-1}"'] \arrow[r, "f'hi_{m-1}"] \arrow[d, "i_{m-1}"] & B' \arrow[d, equal] \\
C' \arrow[d, equal] & X_m \arrow[l, "g'h"'] \arrow[r, "f'h"] \arrow[d, "h"] & B' \arrow[d, equal] \\
C' & A' \arrow[l, "g'"'] \arrow[r, "f'"] & B'.
\end{tikzcd}$$

We number the rows $0, 1, \ldots, m+1$ and let $\widetilde{D}_l$ be the homotopy pushout of the l^{th} row with induced maps $\widetilde{d}_l : \widetilde{D}_l \longrightarrow \widetilde{D}_{l+1}$. Then $\widetilde{D}_0 = \overline{D}$, $\widetilde{D}_{m+1} = D'$ and $d'' = \widetilde{d}_m \cdots \widetilde{d}_1 \widetilde{d}_0$. It suffices to show (i) $L_{\mathcal{A}}(\widetilde{d}_l) \leq 1$ for $l = 0, \ldots, m-1$ and (ii) $L_{\mathcal{A}}(\widetilde{d}_m) = 0$. The argument is similar to (a), and so we content ourselves with noting that

$$\begin{tikzcd}
* \arrow[d] & A_l \arrow[l] \arrow[r] \arrow[d] & * \arrow[d] & & \Sigma A_l \arrow[d] \\
C' \arrow[d, equal] & X_l \arrow[l] \arrow[r] \arrow[d, "i_l"] & B' \arrow[d, equal] \arrow[r, dotted, "\text{homotopy pushout}"] & & \widetilde{D}_l \arrow[d, "\widetilde{d}_l"] \\
C' & X_{l+1} \arrow[l] \arrow[r] & B' & & \widetilde{D}_{l+1}
\end{tikzcd}$$

determines a sequence $\Sigma A_l \longrightarrow \widetilde{D}_l \xrightarrow{\widetilde{d}_l} D_{l+1}$ since ΣA_l is the homotopy pushout of the top row. By the Four Cofibrations Theorem this is a cofiber sequence. Since $\mathcal{A}$ is closed under suspension, $\Sigma A_l \in \mathcal{A}$, and so $L_{\mathcal{A}}(\widetilde{d}_l) \leq 1$. This completes the proof of (1).

To prove (2), we apply Lemma 3.1. Thus, there are commutative diagrams

$$\begin{tikzcd}
& A \arrow[dl, "a"'] \arrow[d, "\alpha"] \arrow[dr, "a"] & \\
A' \arrow[r, "i"] & X \arrow[r, "r"] & A'
\end{tikzcd}
\quad
\begin{tikzcd}
& B \arrow[dl, "b"'] \arrow[d, "\beta"] \arrow[dr, "b"] & \\
B' \arrow[r, "j"] & Y \arrow[r, "s"] & B'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& C \arrow[dl, "c"'] \arrow[d, "\gamma"] \arrow[dr, "c"] & \\
C' \arrow[r, "k"] & Z \arrow[r, "t"] & C'
\end{tikzcd}$$

with $ri = \mathrm{id}$, $sj = \mathrm{id}$, and $tk = \mathrm{id}$, and $L_{\mathcal{A}}(\alpha) = \mathcal{L}_{\mathcal{A}}(a)$, $L_{\mathcal{A}}(\beta) = \mathcal{L}_{\mathcal{A}}(b)$, and

$L_{\mathcal{A}}(\gamma) = \mathcal{L}_{\mathcal{A}}(c)$. Thus we have a diagram

$$\begin{tikzcd}
& C \arrow[ddl, "c"'] \arrow[dd, "\gamma"] \arrow[ddr, "c"] & & & & & & & & & \\
& & & & & A \arrow[ullll, "g"'] \arrow[drrrr, "f"] \arrow[ddl, "a"'] \arrow[dd, "\alpha"] \arrow[ddr, "a"] & & & & & \\
C' \arrow[r, "k"] & Z \arrow[r, "t"] & C' & & & & & & & B \arrow[ddl, "b"'] \arrow[dd, "\beta"] \arrow[ddr, "b"] & \\
& & & & A' \arrow[ullll, "g'"] \arrow[r, "i"] \arrow[drrrr, "f'"'] & X \arrow[ullll, "v"] \arrow[r, "r"] \arrow[drrrr, "u"] & A' \arrow[ullll, "g'"] \arrow[drrrr, "f'"] & & & & \\
& & & & & & & & B' \arrow[r, "j"'] & Y \arrow[r, "s"'] & B'
\end{tikzcd}$$

where $u = jf'r : X \longrightarrow Y$ and $v = kg'r : X \longrightarrow Z$. All triangles and rectangles in the above diagram are commutative. If we denote by E the homotopy pushout of $Z \xleftarrow{v} X \xrightarrow{u} Y$ and the induced maps of homotopy pushouts by $e : D \longrightarrow E$, $l : D' \longrightarrow E$ and $m : E \longrightarrow D'$, then we have a commutative diagram

$$\begin{tikzcd}
& D \arrow[dl, "d"'] \arrow[d, "e"] \arrow[dr, "d"] & \\
D' \arrow[r, "l"] & E \arrow[r, "m"] & D'
\end{tikzcd}$$

with $ml = \mathrm{id}$. Therefore

$$\begin{array}{rcll}
\mathcal{L}_{\mathcal{A}}(d) & \leq & \mathcal{L}_{\mathcal{A}}(e) & \text{since } e \text{ dominates } d \\
& \leq & L_{\mathcal{A}}(e) & \\
& \leq & L_{\mathcal{A}}(\alpha) + \max(L_{\mathcal{A}}(\beta), L_{\mathcal{A}}(\gamma)) & \text{by part (1)} \\
& = & \mathcal{L}_{\mathcal{A}}(a) + \max(\mathcal{L}_{\mathcal{A}}(b), \mathcal{L}_{\mathcal{A}}(c)). &
\end{array}$$

■

Remark 3.3. Our proof of Theorem 3.2 shows that if $\mathcal{A}$ is only assumed to be closed under wedges, then (1) and (2) hold when $A = A'$ and $a = \mathrm{id}$. Moreover the proof also shows that if $\mathcal{A}$ is only assumed to be closed under suspensions, then (1) and (2) hold with $B = B'$, $C = C'$, $b = \mathrm{id}$ and $c = \mathrm{id}$. In Corollary 3.4 below we derive some slightly weaker inequalities than those in Theorem 3.2 that require only closure under suspension.

Corollary 3.4. *Assume the hypotheses of Theorem 3.2, except that $\mathcal{A}$ is not necessarily closed under wedges. Then*

1. $L_{\mathcal{A}}(d) \leq L_{\mathcal{A}}(a) + L_{\mathcal{A}}(b) + L_{\mathcal{A}}(c)$;

2. $\mathcal{L}_{\mathcal{A}}(d) \leq \mathcal{L}_{\mathcal{A}}(a) + \mathcal{L}_{\mathcal{A}}(b) + \mathcal{L}_{\mathcal{A}}(c)$.

The result remains true without assuming that $\mathcal{A}$ is closed under suspensions if $A = A'$ and $a = \mathrm{id}$.

Proof We simply decompose the given map of homotopy pushouts into a composition of three maps:

$$\begin{tikzcd}
C \arrow[d, "c"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, equal] \\
C' \arrow[d, equal] & A \arrow[l, "cg"'] \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, "b"] \\
C' \arrow[d, equal] & A \arrow[l, "cg"'] \arrow[r, "bf"] \arrow[d, "a"] & B' \arrow[d, equal] \\
C' & A' \arrow[l, "g'"'] \arrow[r, "f'"] & B'.
\end{tikzcd}$$

The method of the proof of Theorem 3.2 is then applied to each factor. ∎

4 Applications of the Homotopy Pushout Mapping Theorem

In this section we illustrate the power of the homotopy pushout mapping theorem by obtaining as a consequence a large number of results, some known (in the case $\mathcal{A} = \{\text{all spaces}\}$), and some new.

4.1 Homotopy Pushouts

Corollary 4.1. *Let $\mathcal{A}$ be any collection of spaces. Let*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

be a homotopy pushout square. Then

1. (a) $L_{\mathcal{A}}(B \longrightarrow D) \le L_{\mathcal{A}}(A \longrightarrow C)$,
   (b) $\mathcal{L}_{\mathcal{A}}(B \longrightarrow D) \le \mathcal{L}_{\mathcal{A}}(A \longrightarrow C)$;
2. (a) $\mathrm{cl}_{\mathcal{A}}(D) \le \mathrm{cl}_{\mathcal{A}}(B) + L_{\mathcal{A}}(A \longrightarrow C)$,
   (b) $\mathrm{cat}_{\mathcal{A}}(D) \le \mathrm{cat}_{\mathcal{A}}(B) + \mathcal{L}_{\mathcal{A}}(A \longrightarrow C)$;
3. (a) $\mathrm{kl}_{\mathcal{A}}(B) \le L_{\mathcal{A}}(A \longrightarrow C) + \mathrm{kl}_{\mathcal{A}}(D)$,
   (b) $\mathrm{kit}_{\mathcal{A}}(B) \le \mathcal{L}_{\mathcal{A}}(A \longrightarrow C) + \mathrm{kit}_{\mathcal{A}}(D)$.
4. *If A is closed under wedges, then*
   (a) $L_{\mathcal{A}}(A \longrightarrow D) \le \max(L_{\mathcal{A}}(A \longrightarrow B), L_{\mathcal{A}}(A \longrightarrow C))$,
   (b) $\mathcal{L}_{\mathcal{A}}(A \longrightarrow D) \le \max(\mathcal{L}_{\mathcal{A}}(A \longrightarrow B), \mathcal{L}_{\mathcal{A}}(A \longrightarrow C))$.

Proof The proof of each part amounts to constructing the correct diagram.
Proof of 1 Apply Corollary 3.4 to the diagram

$$\begin{tikzcd}
& A \arrow[rr] \arrow[dl, equal] \arrow[dd, equal] & & B \arrow[dl, equal] \arrow[dd, equal] \\
A \arrow[rr] \arrow[dd] & & B \arrow[dd] & \\
& A \arrow[rr] \arrow[dl] & & B. \arrow[dl] \\
C \arrow[rr] & & D &
\end{tikzcd}$$

Proof of 2 and 3 Apply (1) to the diagram

$$\begin{tikzcd}
& * \arrow[d] \\
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D \arrow[d] \\
& *.
\end{tikzcd}$$

Proof of 4 Map the trivial homotopy pushout diagram

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, equal] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

into the given one, and apply Theorem 3.2. ∎

Corollary 4.2. *Let $\mathcal{A}$ be a collection of spaces that is closed under wedges and suspension and let*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

be a homotopy pushout square. Then

1. *(a)* $\mathrm{cl}_{\mathcal{A}}(D) \leq \mathrm{cl}_{\mathcal{A}}(A) + \max(\mathrm{cl}_{\mathcal{A}}(B), \mathrm{cl}_{\mathcal{A}}(C))$,

   (b) $\mathrm{cat}_{\mathcal{A}}(D) \leq \mathrm{cat}_{\mathcal{A}}(A) + \max(\mathrm{cat}_{\mathcal{A}}(B), \mathrm{cat}_{\mathcal{A}}(C))$*;*

2. *(a)* $\mathrm{kl}_{\mathcal{A}}(D) \leq \mathrm{kl}_{\mathcal{A}}(A) + \max(\mathrm{kl}_{\mathcal{A}}(B), \mathrm{kl}_{\mathcal{A}}(C))$,

   (b) $\mathrm{kit}_{\mathcal{A}}(D) \leq \mathrm{kit}_{\mathcal{A}}(A) + \max(\mathrm{kit}_{\mathcal{A}}(B), \mathrm{kit}_{\mathcal{A}}(C))$.

Proof For (1), apply Theorem 3.2 to the map of the trivial homotopy pushout diagram

$$\begin{array}{ccc} * & = & * \\ \| & & \| \\ * & = & * \end{array}$$

into the given homotopy pushout; for (2), map the given homotopy pushout into the trivial one. ■

Remark 4.3. In the special case $\mathcal{A} = \{\text{all spaces}\}$, Marcum [Mar2] has proved Corollary 4.1(1a) and Hardie [Ha1] has proved Corollary 4.2(1b) (see also [Co3]).

4.2 Mapping Cone Sequences

As noted in §2, a mapping cone sequence $A \longrightarrow B \longrightarrow C$ can be regarded as a homotopy pushout square. Therefore the results of 4.1 apply to mapping cone sequences.

Corollary 4.4. *Let $\mathcal{A}$ be any collection of spaces. Let $A \longrightarrow B \longrightarrow C$ be a mapping cone sequence. Then*

1. *(a)* $\mathrm{cl}_{\mathcal{A}}(C) \le L_{\mathcal{A}}(A \longrightarrow B)$,

   (b) $\mathrm{cat}_{\mathcal{A}}(C) \le \mathcal{L}_{\mathcal{A}}(A \longrightarrow B)$;

2. *(a)* $L_{\mathcal{A}}(B \longrightarrow C) \le \mathrm{kl}_{\mathcal{A}}(A)$,

   (b) $\mathcal{L}_{\mathcal{A}}(B \longrightarrow C) \le \mathrm{kit}_{\mathcal{A}}(A)$;

3. *(a)* $\mathrm{cl}_{\mathcal{A}}(C) \le \mathrm{kl}_{\mathcal{A}}(A) + \mathrm{cl}_{\mathcal{A}}(B)$,

   (b) $\mathrm{cat}_{\mathcal{A}}(C) \le \mathrm{kit}_{\mathcal{A}}(A) + \mathrm{cat}_{\mathcal{A}}(B)$;

4. *(a)* $\mathrm{kl}_{\mathcal{A}}(B) \le \mathrm{kl}_{\mathcal{A}}(A) + \mathrm{kl}_{\mathcal{A}}(C)$,

   (b) $\mathrm{kit}_{\mathcal{A}}(B) \le \mathrm{kit}_{\mathcal{A}}(A) + \mathrm{kit}_{\mathcal{A}}(C)$.

Proof of 1 and 2 Immediate from Corollary 4.1(1).
Proof of 3 and 4 Immediate from (2) and (3) of Corollary 4.1. ■

Remark 4.5. Corollary 4.4(4) shows that $\mathrm{kl}_{\mathcal{A}}$ and $\mathrm{kit}_{\mathcal{A}}$ are subadditive on cofibrations in the following sense (we only state this for $\mathrm{kl}_{\mathcal{A}}$): If $A \longrightarrow X \longrightarrow Q$ is a cofiber sequence, then $\mathrm{kl}_{\mathcal{A}}(X) \le \mathrm{kl}_{\mathcal{A}}(A) + \mathrm{kl}_{\mathcal{A}}(Q)$ (see [A-Str, Thm. 3.4]). This follows (when $\mathcal{A}$ is closed under wedges) since every cofiber sequence is equivalent to a mapping cone sequence. This inequality is not generally true for $\mathrm{cl}_{\mathcal{A}}$ or $\mathrm{cat}_{\mathcal{A}}$ as the cofiber sequence

$$S^2 \longrightarrow \mathbb{C}\mathrm{P}^3 \longrightarrow S^4 \vee S^6$$

shows for the collections $\mathcal{A} = \mathcal{S}$, Σ and $\{\text{all spaces}\}$.

Corollary 4.6. *Let $\mathcal{A}$ be a collection of spaces that is closed under suspension. Consider the map of one mapping cone sequence into another given by the commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C'. \end{tikzcd}$$

Then

1. $L_{\mathcal{A}}(C \longrightarrow C') \leq L_{\mathcal{A}}(A \longrightarrow A') + L_{\mathcal{A}}(B \longrightarrow B')$,
2. $\mathcal{L}_{\mathcal{A}}(C \longrightarrow C') \leq \mathcal{L}_{\mathcal{A}}(A \longrightarrow A') + \mathcal{L}_{\mathcal{A}}(B \longrightarrow B')$.

Proof Apply Corollary 3.4 to the homotopy pushouts obtained from the mapping cone sequences. ∎

4.3 Other Consequences

Corollary 4.7. *Let $\mathcal{A}$ be any collection of spaces. Then for any space B,*

1. $\mathrm{cl}_{\mathcal{A}}(\Sigma B) \leq \mathrm{kl}_{\mathcal{A}}(B)$;
2. $\mathrm{cat}_{\mathcal{A}}(\Sigma B) \leq \mathrm{kit}_{\mathcal{A}}(B)$.

Proof Apply Corollary 4.1(1) to the homotopy pushout square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma B. \end{tikzcd}$$

∎

Corollary 4.8. *Let $\mathcal{A}$ be a collection of spaces that is closed under suspension.*

1. *For any map $f : A \longrightarrow B$,*
   (a) $L_{\mathcal{A}}(f) \leq \mathrm{cl}_{\mathcal{A}}(A) + \mathrm{cl}_{\mathcal{A}}(B)$,
   (b) $\mathcal{L}_{\mathcal{A}}(f) \leq \mathrm{cat}_{\mathcal{A}}(A) + \mathrm{cat}_{\mathcal{A}}(B)$;
2. *For any space A,*
   (a) $\mathrm{kl}_{\mathcal{A}}(A) \leq \mathrm{cl}_{\mathcal{A}}(A)$,
   (b) $\mathrm{kit}_{\mathcal{A}}(A) \leq \mathrm{cat}_{\mathcal{A}}(A)$;
3. *If $f : A \longrightarrow B$ and $g : B \longrightarrow C$, then*
   (a) $L_{\mathcal{A}}(g) \leq L_{\mathcal{A}}(f) + L_{\mathcal{A}}(gf)$,
   (b) $\mathcal{L}_{\mathcal{A}}(g) \leq \mathcal{L}_{\mathcal{A}}(f) + \mathcal{L}_{\mathcal{A}}(gf)$;

4. If $f : A \longrightarrow B$ and $g : B \longrightarrow A$ with $gf = \mathrm{id}$, then

$$\mathcal{L}_{\mathcal{A}}(g) \leq \mathrm{cat}_{\mathcal{A}}(B);$$

5. If $f : A \longrightarrow B$ and $g : B \longrightarrow A$ with $gf = \mathrm{id}$, then

(a) $L_{\mathcal{A}}(g) \leq L_{\mathcal{A}}(f)$,

(b) $\mathcal{L}_{\mathcal{A}}(g) \leq \mathcal{L}_{\mathcal{A}}(f)$.

Proof Again, the proofs depend on finding the appropriate diagram.
Proof of 1 Apply Corollary 3.4 to the diagram

$$\begin{tikzcd}
 & * \arrow[rr, equal] \arrow[dl] \arrow[dd] & & * \arrow[dl] \arrow[dd] \\
A \arrow[rr, equal, crossing over] \arrow[dd, equal] & & A & \\
 & A \arrow[rr] \arrow[dl, equal] & & B. \arrow[dl, equal] \\
A \arrow[rr] & & B \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Proof of 2 Apply (1) to the map $A \longrightarrow *$.
Proof of 3 Apply Corollary 3.4 to the diagram

$$\begin{tikzcd}
 & A \arrow[rr] \arrow[dl, equal] \arrow[dd] & & B \arrow[dl, equal] \arrow[dd, equal] \\
A \arrow[rr, crossing over] \arrow[dd] & & B & \\
 & B \arrow[rr, equal] \arrow[dl] & & B. \arrow[dl] \\
C \arrow[rr, equal] & & C \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Proof of 4 We consider the following mapping of homotopy pushout squares

$$\begin{tikzcd}
 & * \arrow[rr] \arrow[dl] \arrow[dd] & & B \arrow[dl] \arrow[dd, equal] \\
A \arrow[rr, crossing over] \arrow[dd, equal] & & A \vee B & \\
 & B \arrow[rr, equal] \arrow[dl, "g"] & & B. \arrow[dl, "g"] \\
A \arrow[rr, equal] & & A \arrow[from=uu, crossing over, "{(\mathrm{id},g)}"] &
\end{tikzcd}$$

By Corollary 3.4, we immediately conclude that $\mathcal{L}(\mathrm{id}, g) \leq \mathrm{cat}(B)$. Now the commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "g"'] & A \vee B \arrow[r, "(f,\mathrm{id})"] \arrow[d, "(\mathrm{id},g)"] & B \arrow[d, "g"] \\ A \arrow[r, equal] & A \arrow[r, equal] & A \end{tikzcd}$$

shows that g is dominated by (id, g). Thus $\mathcal{L}(g) \leq \mathcal{L}(\mathrm{id}, g) \leq \mathrm{cat}(B)$.
Proof of 5 Apply (3), using the fact that $L(\mathrm{id}) = \mathcal{L}(\mathrm{id}) = 0$.

■

Corollary 4.9. *Let $\mathcal{A}$ be a collection that is closed under suspension and let $f : A \longrightarrow B$. Then*

1. *(a)* $L_{\mathcal{A}}(f) \geq |\mathrm{kl}_{\mathcal{A}}(B) - \mathrm{kl}_{\mathcal{A}}(A)|$,
   (b) $\mathcal{L}_{\mathcal{A}}(f) \geq |\mathrm{kit}_{\mathcal{A}}(B) - \mathrm{kit}_{\mathcal{A}}(A)|$;
2. *(a)* $L_{\mathcal{A}}(f) \geq \mathrm{cl}_{\mathcal{A}}(B) - \mathrm{cl}_{\mathcal{A}}(A)$,
   (b) $\mathcal{L}_{\mathcal{A}}(f) \geq \mathrm{cat}_{\mathcal{A}}(B) - \mathrm{cat}_{\mathcal{A}}(A)$.

Proof We only prove (1a); the other parts are similar. The Composition Axiom, applied to $A \xrightarrow{f} B \longrightarrow *$, implies that

$$\mathrm{kl}_{\mathcal{A}}(A) = L_{\mathcal{A}}(A \longrightarrow *) \leq L_{\mathcal{A}}(B \longrightarrow *) + L_{\mathcal{A}}(f) = \mathrm{kl}_{\mathcal{A}}(B) + L_{\mathcal{A}}(f),$$

so $L_{\mathcal{A}}(f) \geq \mathrm{kl}_{\mathcal{A}}(A) - \mathrm{kl}_{\mathcal{A}}(B)$. On the other hand, Corollary 4.8(3) shows that

$$\mathrm{kl}_{\mathcal{A}}(B) = L_{\mathcal{A}}(B \longrightarrow *) \leq L_{\mathcal{A}}(A \longrightarrow *) + L_{\mathcal{A}}(f) = \mathrm{kl}_{\mathcal{A}}(A) + L_{\mathcal{A}}(f),$$

so $L_{\mathcal{A}}(f) \geq \mathrm{kl}_{\mathcal{A}}(B) - \mathrm{kl}_{\mathcal{A}}(A)$. This proves (1a). ■

Remark 4.10. Assume $\mathcal{A}$ is closed under suspension. Then by Corollary 4.8, $\mathrm{kl}_{\mathcal{A}}(A) \leq \mathrm{cl}_{\mathcal{A}}(A)$ and $\mathrm{kit}_{\mathcal{A}}(A) \leq \mathrm{cat}_{\mathcal{A}}(A)$ (the first inequality was also proved in [A-Str, Thm. 3.3]). Corollary 4.7 then shows that $\mathrm{cl}_{\mathcal{A}}$ and $\mathrm{kl}_{\mathcal{A}}$ agree stably (this was proved by Christensen in [Ch]), and similarly for $\mathrm{cat}_{\mathcal{A}}$ and $\mathrm{kit}_{\mathcal{A}}$. Additionally, Cornea [Co2] has given a completely different proof of Corollary 4.8(4) in the case $\mathcal{A} = \{\text{all spaces}\}$.

4.4 Partial Converse to Theorem 3.2

In this section we show that the formulas of Theorem 3.2 very nearly characterize those collections $\mathcal{A}$ which are closed under wedges or under suspensions.

We introduce the following new construction: for any collection $\mathcal{A}$, The collection $\overline{\mathcal{A}}$ is defined to be

$$\overline{\mathcal{A}} = \{X \mid \mathrm{kl}_{\mathcal{A}}(X) \leq 1\}.$$

Remark 4.11. Clearly $\mathcal{A} \subseteq \overline{\mathcal{A}}$. We note that for certain collections $\mathcal{A}$, $\mathcal{A} \neq \overline{\mathcal{A}}$. We follow the example in Whitehead [Wh, Ex. 3, p. 183]. Let $\mathcal{A}$ be the collection of spheres (or wedges of spheres) and let $X = S^1 \vee S^2$. If $\xi \in \pi_1(X)$ and $\alpha \in \pi_2(X)$ are generators, let $h : S^2 \to X$ be the map whose homotopy class is $2\alpha - \xi \cdot \alpha \in \pi_2(X)$. We set $Y = X \cup_h e^3$, the mapping cone of h, and let $i : S^1 \to Y$ be the inclusion. Then the cofiber of i is contractible, so $\mathrm{kl}_{\mathcal{A}}(Y) \leq 1$. Thus $Y \in \overline{\mathcal{A}}$, but $Y \notin \mathcal{A}$. Note that $\mathcal{A} = \overline{\mathcal{A}}$ if every space in $\mathcal{A}$ is simply-connected.

Our first result shows that passing from $\mathcal{A}$ to $\overline{\mathcal{A}}$ has no effect on the corresponding cone length and category invariants.

Proposition 4.12. *For any map $f : X \longrightarrow Y$,*

1. $L_{\overline{\mathcal{A}}}(f) = L_{\mathcal{A}}(f)$,
2. $\mathcal{L}_{\overline{\mathcal{A}}}(f) = \mathcal{L}_{\mathcal{A}}(f)$.

Proof If suffices to prove (1), because for any collection $\mathcal{A}$, $\mathcal{L}_{\mathcal{A}}(f)$ is the least n for which f is a retract of a map g with $L_{\mathcal{A}}(g) \leq n$ [A-S-S, Prop. 4.3].

Since $\mathcal{A} \subseteq \overline{\mathcal{A}}$, we have $L_{\overline{\mathcal{A}}}(f) \leq L_{\mathcal{A}}(f)$ for any map f, so it remains to prove the reverse inequality. Suppose $L_{\overline{\mathcal{A}}}(f) = n$, and that

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\
X_0 \arrow[r, "j_0"] \arrow[d, equal] & X_1 \arrow[r, "j_1"] & \cdots \arrow[r, "j_{n-2}"] & X_{n-1} \arrow[r, "j_{n-1}"] & X_n \arrow[d, equal] \\
X \arrow[rrrr, "f"] & & & & Y
\end{tikzcd}$$

is a minimal $\overline{\mathcal{A}}$-cone decomposition for f. Thus each $A_i \in \overline{\mathcal{A}}$ and each $A_i \longrightarrow X_i \xrightarrow{j_i} X_{i+1}$ is a mapping cone sequence. Since $A_i \in \overline{\mathcal{A}}$, $\mathrm{kl}_{\mathcal{A}}(A_i) \leq 1$, and hence $L_{\mathcal{A}}(j_i) \leq 1$ by Corollary 4.4(2a). By the Composition Axiom, $L_{\mathcal{A}}(f) \leq n = L_{\overline{\mathcal{A}}}(f)$. ■

We next show that the collection $\overline{\mathcal{A}}$ satisfies the inequality of Theorem 3.2(1) if and only if $\overline{\mathcal{A}}$ is closed under both wedges and suspension. For this it suffices to prove the following corollary.

Corollary 4.13. *Let $\mathcal{A}$ be any collection and consider commutative diagrams of the form*

$$\begin{tikzcd}
C \arrow[d, "c"] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
C' & A' \arrow[l, "g'"] \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

If the inequality

1. $L_{\mathcal{A}}(d) \leq L_{\mathcal{A}}(a) + \max(L_{\mathcal{A}}(b), L_{\mathcal{A}}(c))$

of Theorem 3.2 holds for any such diagram, then $\overline{\mathcal{A}}$ is closed under both wedges and suspension.

Proof We show that $\overline{\mathcal{A}}$ is closed under suspension; the proof of the other assertion is similar. Let $A \in \overline{\mathcal{A}}$ and consider the commutative diagram

$$\begin{tikzcd}
{*} \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & {*} \arrow[d] \\
{*} & {*} \arrow[l] \arrow[r] & {*}.
\end{tikzcd}$$

By (1), we have

$$\begin{array}{rcll}
\mathrm{kl}_{\mathcal{A}}(\Sigma A) & = & L_{\mathcal{A}}(\Sigma A \longrightarrow *) & \\
& \leq & L_{\mathcal{A}}(A \longrightarrow *) & \\
& = & \mathrm{kl}_{\mathcal{A}}(A) & \leq \quad 1,
\end{array}$$

so $\Sigma A \in \overline{\mathcal{A}}$ by definition. ∎

Remark 4.14. To conclude that $\overline{\mathcal{A}}$ is closed under suspension, it suffices to consider only diagrams in which $b = \mathrm{id}$ and $c = \mathrm{id}$, and to conclude that $\overline{\mathcal{A}}$ is closed under wedges, we only need to consider diagrams with $a = \mathrm{id}$.

5 Products

The following is our main result on products of maps.

Theorem 5.1. *Let $\mathcal{A}$ be a collection that is closed under wedges and joins and let $f : A \longrightarrow X$ and $g : B \longrightarrow Y$ be maps. Then*

1. $L_{\mathcal{A}}(f \times g) \leq L_{\mathcal{A}}(f) + L_{\mathcal{A}}(g) + \max(\mathrm{cl}_{\mathcal{A}}(A), \mathrm{cl}_{\mathcal{A}}(B))$,
2. $\mathcal{L}_{\mathcal{A}}(f \times g) \leq \mathcal{L}_{\mathcal{A}}(f) + \mathcal{L}_{\mathcal{A}}(g) + \max(\mathrm{cl}_{\mathcal{A}}(A), \mathrm{cl}_{\mathcal{A}}(B))$.

Proof In the proof of (1) we write $a = \mathrm{cl}_{\mathcal{A}}(A)$, $b = \mathrm{cl}_{\mathcal{A}}(B)$, $m = L_{\mathcal{A}}(f)$ and $n = L_{\mathcal{A}}(g)$ and assume that $a \geq b$.

Now consider the $\mathcal{A}$-cone decompositions of $* \longrightarrow A$ and f

$$\begin{tikzcd}
K_0 \arrow[d] & K_1 \arrow[d] & & K_{a-1} \arrow[d] & K_a \arrow[d] & & K_{m+a-1} \arrow[d] & \\
A_0 \arrow[r] \arrow[d, equal] & A_1 \arrow[r] & \cdots \arrow[r] & A_{a-1} \arrow[r] & A_a \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & A_{m+a-1} \arrow[r] & A_{m+a} \arrow[d, equal] \\
{*} \arrow[rrrr] & & & & A \arrow[rrr, "f"'] & & & X
\end{tikzcd}$$

and of $* \longrightarrow B$ and g

$$\begin{tikzcd}
L_0 \arrow[d] & L_1 \arrow[d] & & L_{b-1} \arrow[d] & L_b \arrow[d] & & L_{m+a-1} \arrow[d] & \\
B_0 \arrow[r] \arrow[d, equal] & B_1 \arrow[r] & \cdots \arrow[r] & B_{b-1} \arrow[r] & B_b \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & B_{n+b-1} \arrow[r] & B_{n+b} \arrow[d, equal] \\
{*} \arrow[rrrr] & & & & B \arrow[rrr, "g"'] & & & Y,
\end{tikzcd}$$

where we identify A with A_a and B with B_b. Since $A_i \subseteq A_{i+1}$ and $B_j \subseteq B_{j+1}$, we may define, for $0 \le k \le n+m+a+b$,

$$C_k = A \times B \ \cup \bigcup_{i+j=k} A_i \times B_j \quad \subseteq \quad A_{m+a} \times B_{n+b}.$$

Observe that $C_b = A \times B$, $C_{n+m+a+b} = X \times Y$ and up to homotopy the composite

$$C_b \longrightarrow C_{b+1} \longrightarrow \cdots \longrightarrow C_{n+m+a+b}$$

is $f \times g$. From this, we see that it suffices to show that $L_{\mathcal{A}}(C_k \longrightarrow C_{k+1}) \le 1$ for each $k \ge b$.

For $0 \le i \le a+m$ and $0 \le j \le b+n$, define

$$P_{ij} = A_i \times B_j, \quad T_{ij} = A_i \times B_{j-1} \cup A_{i-1} \times B_j, \quad \text{and} \quad Q_{ij} = C_{i+j-1} \cup P_{ij}.$$

Then C_{k+1} is obtained as the pushout of all the maps $C_k \longrightarrow Q_{ij}$ with $i+j = k+1$. By an induction based on Corollary 4.1(4) it follows that

$$L_{\mathcal{A}}(C_k \longrightarrow C_{k+1}) \le \max(L_{\mathcal{A}}(C_k \longrightarrow Q_{ij})).$$

Thus, it suffices to show that for $i+j = k+1$, $L_{\mathcal{A}}(C_k \longrightarrow Q_{ij}) \le 1$. Applying Corollary 4.1(1) to the pushout diagram

$$\begin{tikzcd} T_{ij} \arrow[r] \arrow[d] & C_k \arrow[d] \\ P_{ij} \arrow[r] & Q_{ij}, \end{tikzcd}$$

we have $L_{\mathcal{A}}(C_k \longrightarrow Q_{ij}) \le L_{\mathcal{A}}(T_{ij} \longrightarrow P_{ij})$. According to a result of Baues [Bau1] (see also [St2]), there is a mapping cone sequence

$$K_{i-1} * L_{j-1} \longrightarrow T_{ij} \longrightarrow P_{ij}$$

when $i, j > 0$, a mapping cone sequence

$$\begin{tikzcd} L_{j-1} \arrow[r] & T_{0j} \arrow[r] \arrow[d, equal] & P_{0j} \arrow[d, equal] \\ & B_{j-1} & B_j \end{tikzcd}$$

when $i = 0$ and a mapping cone sequence

$$\begin{tikzcd} K_{i-1} \arrow[r] & T_{i0} \arrow[r] \arrow[d, equal] & P_{i0} \arrow[d, equal] \\ & A_{i-1} & A_i \end{tikzcd}$$

when $j = 0$. Since $\mathcal{A}$ is closed under joins, $L_{\mathcal{A}}(T_{ij} \longrightarrow P_{ij}) \le 1$, and this completes the proof of (1).

For (2) we take f' to be a map which dominates f, has the same domain and such that $\mathcal{L}_{\mathcal{A}}(f) = L_{\mathcal{A}}(f')$, and g' is similarly chosen for g (Lemma 3.1). Then (2) is a consequence of (1) since $f' \times g'$ dominates $f \times g$. ■

Corollary 5.2. *If $\mathcal{A}$ is closed under wedges and joins, then*

1. (a) $\mathrm{cl}_{\mathcal{A}}(X \times Y) \le \mathrm{cl}_{\mathcal{A}}(X) + \mathrm{cl}_{\mathcal{A}}(Y)$,
   (b) $\mathrm{kl}_{\mathcal{A}}(X \times Y) \le \mathrm{kl}_{\mathcal{A}}(X) + \mathrm{kl}_{\mathcal{A}}(Y) + \max(\mathrm{cl}_{\mathcal{A}}(X), \mathrm{cl}_{\mathcal{A}}(Y))$;
2. (a) $\mathrm{cat}_{\mathcal{A}}(X \times Y) \le \mathrm{cat}_{\mathcal{A}}(X) + \mathrm{cat}_{\mathcal{A}}(Y)$,
   (b) $\mathrm{kit}_{\mathcal{A}}(X \times Y) \le \mathrm{kit}_{\mathcal{A}}(X) + \mathrm{kit}_{\mathcal{A}}(Y) + \max(\mathrm{cl}_{\mathcal{A}}(X), \mathrm{cl}_{\mathcal{A}}(Y))$.

Remark 5.3. In the case $\mathcal{A} = \{\text{all spaces}\}$, Corollary 5.2(2a) is a classical result due to Bassi [Bas]. Part (1a) has been obtained by Takens [Ta], Clapp and Puppe [C-P2], and Cornea [Co4].

It is possible to improve the inequalities in Corollary 5.2 by imposing stronger conditions on the collection $\mathcal{A}$. To illustrate this, we state and sketch a proof of Proposition 5.4 below. We say that a collection $\mathcal{A}$ is a **$\wedge$-ideal** if for any $A \in \mathcal{A}$ and any space B, the smash product $A \wedge B \in \mathcal{A}$.

Proposition 5.4. *If $\mathcal{A}$ is a $\wedge$-ideal and is closed under wedges and suspensions, then*

1. $\mathrm{kl}_{\mathcal{A}}(X \times Y) \le \mathrm{kl}_{\mathcal{A}}(X) + \mathrm{kl}_{\mathcal{A}}(Y)$ *and*
2. $\mathrm{kit}_{\mathcal{A}}(X \times Y) \le \mathrm{kit}_{\mathcal{A}}(X) + \mathrm{kit}_{\mathcal{A}}(Y)$.

Proof We only prove (1) since the proof of (2) is similar. By applying Corollary 4.4(4) to the sequence $X \vee Y \longrightarrow X \times Y \longrightarrow X \wedge Y$ we conclude that $\mathrm{kl}_{\mathcal{A}}(X \times Y) \le \mathrm{kl}_{\mathcal{A}}(X \vee Y) + \mathrm{kl}_{\mathcal{A}}(X \wedge Y)$. By Corollary 4.2(2), $\mathrm{kl}_{\mathcal{A}}(X \vee Y) \le \max(\mathrm{kl}_{\mathcal{A}}(X), \mathrm{kl}_{\mathcal{A}}(Y))$. Furthermore, a simple argument using the fact that $\mathcal{A}$ is a $\wedge$-ideal shows that $\mathrm{kl}_{\mathcal{A}}(X \wedge Y) \le \min(\mathrm{kl}_{\mathcal{A}}(X), \mathrm{kl}_{\mathcal{A}}(Y))$. This completes the sketch of the proof. ■

6 Pullbacks and Fibrations

We prove a result on pullbacks which yields inequalities for the $\mathcal{A}$-cone length and $\mathcal{A}$-category of the spaces which appear in a fiber sequence.

We begin with a lemma which may be of independent interest. In the proof we denote the half-smash $(X \times Y)/X$ by $X \rtimes Y$ and the quotient map by $q : X \times Y \longrightarrow X \rtimes Y$.

Lemma 6.1. (Cf. [Mar2, Ex. 5.4]) *Let $\mathcal{A}$ be a collection which is closed under joins and let $A \in \mathcal{A}$. If $p_2 : A \times B \longrightarrow B$ is the projection, then*

(1) $L_{\mathcal{A}}(p_2) \le \mathrm{cl}_{\mathcal{A}}(B) + 1$

(2) $\mathcal{L}_{\mathcal{A}}(p_2) \le \mathrm{cat}_{\mathcal{A}}(B) + 1$.

Proof Consider the map $p : A \rtimes B \longrightarrow B$ induced by p_2. The main step in the proof is to show

$$L_{\mathcal{A}}(p) \leq \mathrm{cl}_{\mathcal{A}}(B) \qquad \text{and} \qquad \mathcal{L}_{\mathcal{A}}(p) \leq \mathrm{cat}_{\mathcal{A}}(B).$$

Suppose we have a diagram:

$$\begin{tikzcd}
 & L_0 \arrow[d] & L_1 \arrow[d] & & L_{n-1} \arrow[d] & \\
* = & B_0 \arrow[r, "j_0"] & B_1 \arrow[r, "j_1"] & \cdots \arrow[r] & B_{n-1} \arrow[r, "j_{n-1}"] & B_n
\end{tikzcd}$$

and a map $f_n : B_n \longrightarrow B$ with $L_i \in A$. Define D_i as the homotopy pushout in the diagram

$$\begin{tikzcd}
A \rtimes B_i \arrow[r, "q_i"] \arrow[d, "\mathrm{id} \rtimes f_n j_{n-1} \ldots j_i"'] & B_i \arrow[d, "r_i"] \\
A \rtimes B \arrow[r, "s_i"] & D_i,
\end{tikzcd}$$

where q_i is the projection. Then there are maps $k_i : D_i \longrightarrow D_{i+1}$ with $k_i s_i = s_{i+1}$. When $i = 0$ we have $D_0 = A \rtimes B$ and when $i = n$ we have

$$\begin{tikzcd}
A \rtimes B_n \arrow[r, "q_n"] \arrow[d, "\mathrm{id} \rtimes f_n"'] & B_n \arrow[d, "r_n"] \\
A \rtimes B \arrow[r, "s_n"] & D_n.
\end{tikzcd}$$

From the above diagram and the maps

$$f_n : B_n \longrightarrow B \quad \text{and} \quad p : A \rtimes B \longrightarrow B,$$

we obtain a map $g_n : D_n \longrightarrow B$ such that $g_n s_n = p$ and $g_n r_n = f_n$. It then follows that

$$g_n k_{n-1} \cdots k_0 = p.$$

Now we prove (1). Suppose f_n is a homotopy equivalence so our given decomposition is an $\mathcal{A}$-cone decomposition of B of length n. By the previous homotopy pushout diagram, r_n is a homotopy equivalence and from $g_n r_n = f_n$ we obtain that g_n is a homotopy equivalence. Since $g_n k_{n-1} \ldots k_0 = p$, we get $L_A(p) \leq L_{\mathcal{A}}(k_0) + \cdots + L_{\mathcal{A}}(k_{n-1})$. To complete the proof that $L_{\mathcal{A}}(p) \leq n = \mathrm{cl}_{\mathcal{A}}(B)$, it suffices to show that $L_{\mathcal{A}}(k_i) \leq 1$. But $L_i \longrightarrow B_i \xrightarrow{j_i} B_{i+1}$ is a mapping cone sequence and so

$$A \rtimes L_i \longrightarrow A \rtimes B_i \longrightarrow A \rtimes B_{i+1}$$

is also a mapping cone sequence. Thus we have a commutative diagram

$$\begin{tikzcd}
* \arrow[d] & A \rtimes L_i \arrow[l] \arrow[r] \arrow[d] & L_i \arrow[d] \\
A \rtimes B \arrow[d] & A \rtimes B_i \arrow[l] \arrow[r, "q_i"] \arrow[d] & B_i \arrow[d] \\
A \rtimes B & A \rtimes B_{i+1} \arrow[l] \arrow[r, "q_{i+1}"] & B_{i+1}.
\end{tikzcd}$$

Since each column is a cofiber sequence, $P \longrightarrow D_i \xrightarrow{k_i} D_{i+1}$ is a cofiber sequence by the Four Cofibrations Theorem, where P is the homotopy pushout of the top line. However it is easily seen that $P = A * L_i$, the join of A and L_i. But $P \in \mathcal{A}$ since $\mathcal{A}$ is closed under joins. Thus $L_{\mathcal{A}}(k_i) \leq 1$ and so $L_{\mathcal{A}}(p) \leq n = \mathrm{cl}_{\mathcal{A}}(B)$. Part 1 of the lemma now follows by factoring $p_2 : A \times B \longrightarrow B$ as

$$A \times B \xrightarrow{q} A \rtimes B \xrightarrow{p} B.$$

Since $A \longrightarrow A \times B \xrightarrow{q} A \rtimes B$ is mapping cone sequence with $A \in \mathcal{A}$, $L_{\mathcal{A}}(q) \leq 1$. Thus $L_{\mathcal{A}}(p_2) \leq L_{\mathcal{A}}(p) + L_{\mathcal{A}}(q) \leq \mathrm{cl}_{\mathcal{A}}(B) + 1$ by the Composition Axiom.

The proof of (2) is similar. Instead of taking an $\mathcal{A}$-cone decomposition of B, we take an $\mathcal{A}$-category decomposition of B of length n. Thus instead of having $f_n : B_n \longrightarrow B$ a homotopy equivalence, we have a map $s : B \longrightarrow B_n$ with $f_n s \simeq \mathrm{id}$. We define $\sigma : B \longrightarrow D_n$ by $\sigma = r_n s$. Then the following are easily checked:

$$(a) \quad g_n\sigma \simeq \mathrm{id} \qquad (b) \quad \sigma p \simeq k_{n-1} \cdots k_0 \qquad (c) \quad g_n k_{n-1} \cdots k_0 \simeq p.$$

Using the maps (id, σ) and (id, g_n) we see that p is homotopy dominated by $k_{n-1} \cdots k_0$. Therefore $\mathcal{L}_{\mathcal{A}}(p) \leq \mathcal{L}_{\mathcal{A}}(k_0) + \cdots + \mathcal{L}_{\mathcal{A}}(k_{n-1})$. The rest of the proof is the same as the proof of (1), using $\mathcal{L}$ for L. ■

Now we prove our pullback theorem.

Theorem 6.2. *Let $\mathcal{A}$ be a collection that is closed under wedges and joins and let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

be a pullback diagram. Let $B \longrightarrow D$ be a fibration with fiber F. Then

1. $L_{\mathcal{A}}(A \longrightarrow B) \leq L_{\mathcal{A}}(C \longrightarrow D)(\mathrm{cl}_{\mathcal{A}}(F) + 1)$;
2. $\mathcal{L}_{\mathcal{A}}(A \longrightarrow B) \leq \mathcal{L}_{\mathcal{A}}(C \longrightarrow D)(\mathrm{cat}_{\mathcal{A}}(F) + 1)$.

Proof We prove (1). Let

$$\begin{tikzcd} K_0 \arrow[d] & K_1 \arrow[d] & & K_{n-1} \arrow[d] & \\ C_0 \arrow[r] \arrow[d, equal] & C_1 \arrow[r] & \cdots \arrow[r] & C_{n-1} \arrow[r] & C_n \arrow[d, equal] \\ C \arrow[rrrr] & & & & D \end{tikzcd}$$

be a minimal $\mathcal{A}$-cone decomposition for $C \longrightarrow D$. For $0 \leq i \leq n$, define B_i to be the pullback indicated by the square

$$\begin{tikzcd} B_i \arrow[r] \arrow[d] & B \arrow[d] \\ C_i \arrow[r] & D. \end{tikzcd}$$

Thus $B_0 = A$, $B_n \equiv B$ and we obtain maps $B_i \longrightarrow B_{i+1}$. With these identifications, the composition $B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_n$ is simply $A \longrightarrow B$. Hence, it suffices to show that $L(B_i \longrightarrow B_{i+1}) \leq \mathrm{cl}_{\mathcal{A}}(F) + 1$. Consider the cube diagram

$$\begin{tikzcd}
 & K_i \times F \arrow[rr] \arrow[dl] \arrow[dd] & & F \arrow[dl] \arrow[dd] \\
B_i \arrow[rr] \arrow[dd] & & B_{i+1} \arrow[dd] & \\
 & K_i \arrow[rr] \arrow[dl] & & * \arrow[dl] \\
C_i \arrow[rr] & & C_{i+1}. &
\end{tikzcd}$$

In this diagram, the bottom square is a homotopy pushout and the sides are pullbacks. This assertion is obvious for all squares except the left side square

$$\begin{tikzcd}
K_i \times F \arrow[r] \arrow[d] & B_i \arrow[d] \\
K_i \arrow[r] & C_i.
\end{tikzcd}$$

To see that this is a pullback square, let P be the pullback of $B_i \longrightarrow C_i \longleftarrow K_i$. Then P is also the pullback of $B_{i+1} \longrightarrow C_{i+1} \longleftarrow C_i \longleftarrow K_i$. Since the composite $C_{i+1} \longleftarrow C_i \longleftarrow K_i$ is the constant map, the latter pullback is $K_i \times F$.

Now by the Mather's second cube theorem [Mat], the top square is a homotopy pushout. By Corollary 4.1(1), $L_{\mathcal{A}}(B_i \longrightarrow B_{i+1}) \leq L_{\mathcal{A}}(K_i \times F \longrightarrow F)$. By Lemma 6.1(1), $L_{\mathcal{A}}(K_i \times F \longrightarrow F) \leq \mathrm{cl}_{\mathcal{A}}(F) + 1$. This proves (1).

The proof of (2) is similar and uses Lemma 6.1(2) and we omit it. ∎

Corollary 6.3. *Let $\mathcal{A}$ be a collection that is closed under wedges and joins and let $F \longrightarrow E \longrightarrow B$ be a fibration. Then*

1. $\mathrm{cl}_{\mathcal{A}}(E) + 1 \leq (\mathrm{cl}_{\mathcal{A}}(B) + 1)(\mathrm{cl}_{\mathcal{A}}(F) + 1)$*;*
2. $\mathrm{cat}_{\mathcal{A}}(E) + 1 \leq (\mathrm{cat}_{\mathcal{A}}(B) + 1)(\mathrm{cat}_{\mathcal{A}}(F) + 1)$*.*

Proof We prove (1). Applying Theorem 6.2 to the pullback square

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[d] \\
* \arrow[r] & B,
\end{tikzcd}$$

we obtain $L_{\mathcal{A}}(F \longrightarrow E) \leq \mathrm{cl}_{\mathcal{A}}(B)(\mathrm{cl}_{\mathcal{A}}(F) + 1)$. Now the Composition Axiom shows that

$$\mathrm{cl}_{\mathcal{A}}(E) \leq \mathrm{cl}_{\mathcal{A}}(F) + L_{\mathcal{A}}(F \longrightarrow E),$$

so

$$\begin{aligned}
\mathrm{cl}_{\mathcal{A}}(E) + 1 &\leq \mathrm{cl}_{\mathcal{A}}(F) + L_{\mathcal{A}}(F \longrightarrow E) + 1 \\
&\leq \mathrm{cl}_{\mathcal{A}}(B)(\mathrm{cl}_{\mathcal{A}}(F) + 1) + (\mathrm{cl}_{\mathcal{A}}(F) + 1) \\
&= (\mathrm{cl}_{\mathcal{A}}(B) + 1)(\mathrm{cl}_{\mathcal{A}}(F) + 1).
\end{aligned}$$

∎

Remark 6.4. In the special case $\mathcal{A} = \{\text{all spaces}\}$ we retrieve Varadarajan's result [Var]

$$\mathrm{cat}(E) + 1 \leq (\mathrm{cat}(B) + 1)(\mathrm{cat}(F) + 1).$$

Hardie has obtained a further improvement in [Ha2], but that involves a different notion of the category of a map from the one we consider here [B-G, Fo].

7 Miscellaneous Results and Problems

In this section we consider several topics. We first establish some elementary, but useful, facts about $L_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}}$. We then show that some known results for the collection $\mathcal{A} = \{\text{all spaces}\}$ do not hold for an arbitrary collection $\mathcal{A}$. Finally, we conclude the section by stating a number of open questions and discussing them briefly.

We begin with a few elementary results.

Proposition 7.1. *Let $f : X \longrightarrow Y$ and let $\mathcal{A}$ be any collection. Then*

1. *$L_{\mathcal{A}}(f) = 0$ if and only if f is a homotopy equivalence;*

2. *$\mathcal{L}_{\mathcal{A}}(f) = 0$ if and only if f is a homotopy equivalence.*

Proof We prove (1) and (2) at the same time. By the axioms, if f is a homotopy equivalence, then $\mathcal{L}_{\mathcal{A}}(f) = L_{\mathcal{A}}(f) = 0$. Conversely, define a function $\ell_{\mathcal{A}}$ by

$$\ell_{\mathcal{A}}(f) = \begin{cases} 0 & \text{if } f \text{ is a homotopy equivalence} \\ 1 & \text{otherwise.} \end{cases}$$

It is trivial to check that $\ell_{\mathcal{A}}$ satisfies the $\mathcal{A}$-category axioms, so

$$\ell_{\mathcal{A}}(f) \leq \mathcal{L}_{\mathcal{A}}(f) \leq L_{\mathcal{A}}(f)$$

for every map f. Consequently, if f is not a homotopy equivalence, then $L_{\mathcal{A}}(f) \geq \mathcal{L}_{\mathcal{A}}(f) \geq \ell_{\mathcal{A}}(f) = 1$. ■

Proposition 7.2. *Let $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y'$ be maps and let $\mathcal{A}$ be a collection that is closed under wedges. Then*

(a) $L_{\mathcal{A}}(f \vee g) \leq \max(L_{\mathcal{A}}(f), L_{\mathcal{A}}(g))$;

(b) $\mathcal{L}_{\mathcal{A}}(f \vee g) = \max(\mathcal{L}_{\mathcal{A}}(f), \mathcal{L}_{\mathcal{A}}(g))$.

Proof Since both $X \vee X'$ and $Y \vee Y'$ are homotopy pushouts, the inequality $\mathcal{L}_{\mathcal{A}}(f \vee g) \leq \max(\mathcal{L}_{\mathcal{A}}(f), \mathcal{L}_{\mathcal{A}}(g))$ is a consequence of Theorem 3.2. This same argument shows $L_{\mathcal{A}}(f \vee g) \leq \max(L_{\mathcal{A}}(f), L_{\mathcal{A}}(g))$. The reverse inequality for $\mathcal{L}_{\mathcal{A}}(f \vee g)$ follows since f and g are both retracts of $f \vee g$. ■

An example due to Dupont [Du] can be used to show that equality does not generally hold in (a). Other such examples can be found in [St1], where spaces X_n with category n and cone length $n+1$ are constructed. According to an observation of Ganea [Ta] (see also [Co2]), this implies that there is a space A such that $\mathrm{cl}(X_n \vee \Sigma A) = \mathrm{cat}(X_n)$. If we let $\mathcal{A} = \{\text{all spaces}\}$, $f : * \longrightarrow X_n$ and $g : * \longrightarrow \Sigma A$, then we have

$$L_{\mathcal{A}}(f) = \mathrm{cl}(X_n) > \mathrm{cat}(X_n) = \mathrm{cl}(X_n \vee \Sigma A) = L_{\mathcal{A}}(f \vee g).$$

Corollary 7.3. *Let X and Y be spaces and $\mathcal{A}$ a collection that is closed under wedges. Then*

(a) $L_{\mathcal{A}}(X \xrightarrow{*} Y) \leq \max(\mathrm{kl}_{\mathcal{A}}(X), \mathrm{cl}_{\mathcal{A}}(Y))$*;*

(b) $\mathcal{L}_{\mathcal{A}}(X \xrightarrow{*} Y) = \max(\mathrm{kit}_{\mathcal{A}}(X), \mathrm{cat}_{\mathcal{A}}(Y))$.

Proof The trivial map $X \xrightarrow{*} Y$ is the wedge of the maps $X \longrightarrow *$ and $* \longrightarrow Y$. The result follows from Proposition 7.2. ∎

Next we turn to some known results for the collection $\mathcal{A} = \{\text{all spaces}\}$. For this collection we delete the subscript $\mathcal{A}$ and write $L_{\mathcal{A}}$ as L, $\mathcal{L}_{\mathcal{A}}$ as $\mathcal{L}$, etc.

For any map $f : X \longrightarrow Y$, it has been proved that $L(f) \leq \mathrm{cl}(Y) + 1$ [Mar2] and $\mathcal{L}(f) \leq \mathrm{cat}(Y) + 1$ [Co2]. We show that this may not be true for an arbitrary collection $\mathcal{A}$.

Example 7.4. By Corollary 4.9, $\mathrm{kl}_{\mathcal{A}}(X) \leq L_{\mathcal{A}}(f) + \mathrm{kl}_{\mathcal{A}}(Y)$. Thus if $L_{\mathcal{A}}(f) \leq \mathrm{cl}_{\mathcal{A}}(Y) + 1$ were true, we would have

$$\mathrm{kl}_{\mathcal{A}}(X) \leq \mathrm{cl}_{\mathcal{A}}(Y) + \mathrm{kl}_{\mathcal{A}}(Y) + 1,$$

for *any* X and Y. This cannot hold for any collection $\mathcal{A}$ such that there are spaces X with arbitrarily large killing length (e.g., for $\mathcal{A} = \mathcal{S}$ or Σ). The analogous observation holds for $\mathcal{L}_{\mathcal{A}}$.

Another classical result concerns the homotopy pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D. \end{tikzcd}$$

It has been shown that $\mathrm{cl}(D) \leq \mathrm{cl}(B) + \mathrm{cl}(C) + 1$ [Ha1]. We show that this is not true for $\mathcal{A} = \mathcal{S}$, the collection of wedges of spheres.

Example 7.5. Consider the homotopy pushout

$$\begin{tikzcd} \mathbb{C}\mathrm{P}^t \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma\mathbb{C}\mathrm{P}^t. \end{tikzcd}$$

As t increases, the length of the longest nontrivial composition of Steenrod squares in $H^*(\Sigma\mathbb{C}\mathrm{P}^t; \mathbb{Z}_2)$ also becomes arbitrarily large. It follows from [A-S-S, Prop. 7.5] that $\mathrm{cl}_{\mathcal{S}}(\Sigma\mathbb{C}\mathrm{P}^t)$ increases as t increases. This contradicts the $\mathcal{S}$-analog of Hardie's result.

We conclude the paper by stating and discussing three open problems.

Problem 7.6 We have seen in [A-S-S, Prop. 7.3] that for certain collections $\mathcal{A}$, $\mathrm{wcat}(X) \le 2^{\mathrm{kl}_{\mathcal{A}}(X)} - 1$, where $\mathrm{wcat}(X)$ is the weak category of X (see [Gi, Ja]). Since $\mathrm{wcat}(X) \le \mathrm{cat}(X) \le \mathrm{cat}_{\mathcal{A}}(X)$ for any collection $\mathcal{A}$, it is reasonable to ask for which collections $\mathcal{A}$ is $\mathrm{cat}_{\mathcal{A}}(X) \le 2^{\mathrm{kl}_{\mathcal{A}}(X)} - 1$. Of course $\mathcal{A}$ must not be {all spaces}, since $\mathrm{kl}_{\mathcal{A}}(X) \le 1$ for every space X in that case. We note that the conjecture has been verified in the case $\mathcal{A} = \mathcal{S}$ and $X = S_1^n \times \cdots \times S_r^n$ [A-M-S, Prop. 6.2]. Other evidence for the conjecture in the case $\mathcal{A} = \Sigma$, the collection of suspensions, has been given in [A-Str], where a weaker form of this problem has been posed [A-Str, §7, No. 5].

Problem 7.7 Given $f : X \longrightarrow Y$. Is $L_{\mathcal{A}}(f) \le \mathrm{kl}_{\mathcal{A}}(X) + \mathrm{cl}_{\mathcal{A}}(Y)$, and is $\mathcal{L}_{\mathcal{A}}(f) \le \mathrm{kit}_{\mathcal{A}}(X) + \mathrm{cat}_{\mathcal{A}}(Y)$?

We discuss the evidence in the case of $L_{\mathcal{A}}$ (the discussion is analogous for $\mathcal{L}_{\mathcal{A}}$). First of all, if C is the cofiber of f, it is true that $\mathrm{cl}_{\mathcal{A}}(C) \le \mathrm{kl}_{\mathcal{A}}(X) + \mathrm{cl}_{\mathcal{A}}(Y)$ (Corollary 4.4(3a)) and also $\mathrm{cl}_{\mathcal{A}}(C) \le L_{\mathcal{A}}(f)$ (Corollary 4.4(1a)). Secondly, we have that $L_{\mathcal{A}}(f) \le \mathrm{cl}_{\mathcal{A}}(X) + \mathrm{cl}_{\mathcal{A}}(Y)$ (Corollary 4.8(1a)) and $\mathrm{kl}_{\mathcal{A}}(X) \le \mathrm{cl}_{\mathcal{A}}(X)$. Finally, when $\mathcal{A} =$ {all spaces} then $\mathrm{kl}_{\mathcal{A}}(X) = 1$ for every X, and in this case it is known that $L(f) \le \mathrm{cl}(Y) + 1$ [Mar2].

Problem 7.8 It is well known that $\mathrm{cl}(X) \le \mathrm{cat}(X) + 1$ [Ta]. If $\mathcal{A}$ is a collection different from {all spaces}, is there an upper bound for $\mathrm{cl}_{\mathcal{A}}(X)$ in terms of $\mathrm{cat}_{\mathcal{A}}(X)$? This question was asked by Scheerer-Tanré in [S-T]. Analogously, is there an upper bound for $\mathrm{kl}_{\mathcal{A}}$ in terms of $\mathrm{kit}_{\mathcal{A}}$?

We can show that $\mathrm{kit}_{\Sigma}(X) \le 1$ implies $\mathrm{kl}_{\Sigma}(X) \le 3$ as follows. If $\mathrm{kit}_{\Sigma}(X) \le 1$ then there is a mapping cone sequence $A \longrightarrow X \xrightarrow{*} Y$ with $A \in \Sigma$. It follows that $\Sigma A = Y \vee \Sigma X$, and so there is a retraction map $\alpha : \Sigma A \longrightarrow Y$. The cofiber of α is $\Sigma^2 X$, and hence we have a decomposition

$$\begin{tikzcd} A \arrow[d] & \Sigma A \arrow[d] & \Sigma^2 X \arrow[d] & \\ X \arrow[r] & Y \arrow[r] & \Sigma^2 X \arrow[r] & *. \end{tikzcd}$$

This proves that $\mathrm{kl}_{\Sigma}(X) \le 3$.

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