Euler's constants for the Selberg and the Dedekind zeta functions

Yasufumi Hashimoto Yasuyuki Iijima Nobushige Kurokawa*
Masato Wakayama†

Abstract

The purpose of this paper is to study an analogue of Euler's constant for the Selberg zeta functions of a compact Riemann surface and the Dedekind zeta function of an algebraic number field. Especially, we establish similar expressions of such Euler's constants as de la Vallée-Poussin obtained in 1896 for the Riemann zeta function. We also discuss, so to speak, higher Euler's constants and establish certain formulas concerning the power sums of essential zeroes of these zeta functions similar to Riemann's explicit formula.

1 Introduction

Recall first the Riemann zeta function

$$\zeta(s) = \prod_{p:\text{prime}} \left(1 - p^{-s}\right)^{-1}.$$

Around 1859, Riemann discovered the following identity

$$\begin{aligned} \sum_{\rho \in Z} \frac{1}{\rho} &= 1 + \frac{\gamma}{2} - \frac{1}{2}\log\pi - \log 2 \qquad (1.1) \\ &= 0.0230957089661210338\cdots, \end{aligned}$$

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where ρ runs over the set Z of the essential zeroes (*i.e.*, $0 < \mathrm{Re}(\rho) < 1$) of $\zeta(s)$ counting with a possible multiplicity, and γ is the Euler constant given by

$$\begin{aligned}\gamma &= \lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right)\\ &= \lim_{x\to\infty}\left(\sum_{n<x}\frac{1}{n} - \log x\right) = 0.57721566490153286060\cdots.\end{aligned}$$

Note also that the information of the equation (1.1) is contained in the following factorization.

$$\zeta(s) = e^{(\log(2\pi)-1-\frac{\gamma}{2})s}\frac{1}{2(s-1)\Gamma(1+\frac{s}{2})}\prod_{\rho\in Z}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}. \tag{1.2}$$

Riemann used (1.1) [the above numerical computation is due to Riemann] to calculate essential zeros and verified his famous conjecture, Riemann hypothesis [R] for several zeroes ; for example, he obtained that the first essential zero is $\rho_1 \approx \frac{1}{2} + 14.14i$. Riemann's studies on the essential zeroes were written in unpublished manuscripts, and some details were investigated by Siegel [Sie].

Riemann's identity (1.1) is the first example of the so called explicit formulas for zeta functions. Zeta functions are in general defined as Euler products over (generalized) primes, and so called explicit/trace formulas describe relations between these (generalized) primes and zeroes (and poles) of zeta functions. The nature of Riemann's identity (1.1) becomes clear as a relation between zeroes and primes when we recall the identity

$$\gamma = \lim_{x\to\infty}\left(\log x - \sum_{\substack{p<x\\ p:\text{prime}}}\frac{\log p}{p-1}\right) = \lim_{x\to\infty}\left(\log x - \sum_{n<x}\frac{\Lambda(n)}{n}\right) \tag{1.3}$$

of de la Vallée-Poussin [VP] in 1896, where $\Lambda(n)$ is the von Mangoldt function given by

$$\Lambda(n) = \begin{cases}\log p & \text{if } n = p^{\ell} \text{ for a prime } p \text{ and a positive integer } \ell\\ 0 & \text{otherwise.}\end{cases}$$

Note also that de la Vallée-Poussin proved (1.3) together with the prime number theorem.

There is a similar situation for the Selberg zeta function (see [Sel]). Actually it describes the prime geodesic theorem for a Riemann surface. In order to make things more explicit, let us consider a discrete co-compact torsion free subgroup Γ of $SL(2,\mathbb{R})$. Let $H = SL(2,\mathbb{R})/SO(2) = \{z = x+iy\,;\, x,y\in\mathbb{R},\ y>0\}$ be the upper half plane. Then the quotient $\Gamma\backslash H$ forms a Riemann surface of genus $g>1$ with area $\mu(\Gamma\backslash H) = 4\pi(g-1)$. The Selberg zeta function $Z_\Gamma(s)$ defined by

$$Z_\Gamma(s) = \prod_{P\in\mathrm{Prim}(\Gamma)}\prod_{n=0}^{\infty}(1-N(P)^{-s-n}) \qquad \mathrm{Re}(s)>1,$$

where $\mathrm{Prim}(\Gamma)$ is the set of primitive hyperbolic conjugacy classes of Γ, and $N(P)$ denotes the norm of P, that is, the square of the larger eigenvalue of a representative

2×2 matrix of P. Here a hyperbolic element P (and hence also its conjugacy class) is said to be primitive when P is a generator of an infinite cyclic group $\mathcal{Z}_\Gamma(P)$, the centralizer of P in Γ. Hence the every hyperbolic element (class) γ can be uniquely written as $\gamma = P_\gamma^\ell$ for some $P_\gamma \in \mathrm{Prim}(\Gamma)$ and a positive integer ℓ. We denote also by $\mathrm{Hyp}(\Gamma)$ the set of hyperbolic conjugacy classes of Γ.

It is well-known that $Z_\Gamma(s)$ is continued analytically to the whole plane $\mathbb{C}$ as an entire function of order 2 via the following equation for the logarithmic derivative of $Z_\Gamma(s)$, which is got by the trace formula;

$$\begin{aligned}\frac{1}{2s-1}\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} &= \frac{1}{2\beta}\frac{Z_\Gamma'(\frac12+\beta)}{Z_\Gamma(\frac12+\beta)} + \sum_{n=0}^\infty\left\{\frac{1}{r_n^2+(s-\frac12)^2} - \frac{1}{r_n^2+\beta^2}\right\} \\ &\quad + 2(g-1)\sum_{k=0}^\infty\left\{\frac{1}{k+\frac12+\beta} - \frac{1}{k+s}\right\}, \end{aligned} \tag{1.4}$$

where $\mathrm{Re}(\beta) > 1$, and $\{\frac14 + r_n^2\}_{n=0,1,2,\ldots}$ are the eigenvalues of the Laplacian $\Delta_\Gamma = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ on the Riemann surface $\Gamma\backslash H$. We put $\lambda_n = \frac14 + r_n^2$ and arrange the order of λ_n by $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, each counting with multiplicity m_n. Then $Z_\Gamma(s)$ has essential (non-trivial) zeroes at $s = \frac12 \pm ir_n$ of order m_n. Since the multiplicity m_0 of the eigenvalue $\lambda_0 = 0$ is equal to 1, it follows from (1.4) that the logarithmic derivative of $Z_\Gamma(s)$ has a pole at 1 with residue 1, whence $Z_\Gamma(s)$ has a simple zero at $s = 1$. Let

$$Z_\Gamma(s) = \alpha_1(\Gamma)(s-1) + \alpha_2(\Gamma)(s-1)^2 + \alpha_3(\Gamma)(s-1)^3 + \cdots \tag{1.5}$$

be the Taylor expansion around $s = 1$. Also, the logarithmic derivative of $Z_\Gamma(s)$ is written as

$$\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} = \frac{1}{s-1} + \gamma_\Gamma^{(0)} + \gamma_\Gamma^{(1)}(s-1) + \gamma_\Gamma^{(2)}(s-1)^2 + \cdots, \tag{1.6}$$

around $s = 1$. The following relations are clearly examined by comparing these two expansions and from the expression (1.4) above.

$$\gamma_\Gamma^{(0)} = \frac{\alpha_2(\Gamma)}{\alpha_1(\Gamma)} = \lim_{s\to1}\left(\frac{1}{2s-1}\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} - \frac{1}{s(s-1)}\right) = \frac12\frac{Z_\Gamma''(1)}{Z_\Gamma'(1)} \tag{1.7}$$

We call this $\gamma_\Gamma^{(0)}$ the Euler-Selberg constant for Γ. The Euler-Selberg constant has been appeared in [KW1]. It is also immediate to see the relations

$$\gamma_\Gamma^{(1)} = -\frac{\alpha_2(\Gamma)^2}{\alpha_1(\Gamma)^2} + 2\frac{\alpha_3(\Gamma)}{\alpha_1(\Gamma)}, \quad \gamma_\Gamma^{(2)} = 2\frac{\alpha_2(\Gamma)^3}{\alpha_1(\Gamma)^3} - 3\frac{\alpha_2(\Gamma)\alpha_3(\Gamma)}{\alpha_1(\Gamma)^2} + 3\frac{\alpha_4(\Gamma)}{\alpha_1(\Gamma)}, \ldots.$$

Moreover the factorization formula for $Z_\Gamma(s)$ which is a close analogue of (1.2) is known as follows.

$$\begin{aligned} Z_\Gamma(s) &= Z_\Gamma'(1)s(s-1)e^{\{\gamma_\Gamma^{(0)}+2(g-1)\gamma\}s(s-1)}[(2\pi)^{s-1}\Gamma_2(s)\Gamma_2(s+1)]^{-2(g-1)} \\ &\quad \times \prod_{n=1}^\infty\left(1 + \frac{s(s-1)}{\lambda_n}\right)\exp\left(-\frac{s(s-1)}{\lambda_n}\right), \end{aligned}$$

where $\Gamma_2(s)$ denotes the double Gamma function. See [Ste], [Vor] for details. Furthermore, note that $Z_\Gamma(s)$ satisfies the following functional equation;

$$Z_\Gamma(s) = Z_\Gamma(1-s)\exp\left(4\pi(g-1)\int_0^{s-\frac{1}{2}} r\tan(\pi r)dr\right). \tag{1.8}$$

If we set $\hat{Z}_\Gamma(s) = \{\Gamma_2(s)\Gamma_2(s+1)\}^{1-g}Z_\Gamma(s)$, it is known that the functional equation (1.8) can be recast into the simpler form $\hat{Z}_\Gamma(s) = \hat{Z}_\Gamma(1-s)$.

The purpose of this paper is to generalize the Euler constant and describe the power of essential zeroes such as (1.1), in more general situations. Especially, we generalize (1.1) and (1.3) to the case of the Selberg zeta functions for compact Riemann surfaces and the case of the Dedekind zeta function of algebraic number fields.

We have first the following formulas of the Euler-Selberg constant and the analogue of (1.1) for the essential zeroes of the Selberg zeta function.

Theorem A. *Let Γ be a discrete co-compact torsion free subgroup of $SL(2,\mathbb{R})$. For $\gamma \in \mathrm{Hyp}(\Gamma)$, we denote by P_γ the primitive hyperbolic class class such that $\gamma \in \mathcal{Z}_\Gamma(P_\gamma)$. Then we have*

(1) $$\gamma_\Gamma^{(0)} = \lim_{x\to\infty}\left\{\sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1} - \log x\right\}.$$

(2) $$\sum_{n>0}\frac{1}{\lambda_n^2} = 2\gamma_\Gamma^{(0)} - \gamma_\Gamma^{(1)} + (g-1)\frac{\pi^2}{3} - 3.$$

Remark 1.1. The assertion (2) above shows that it is preferable to get similar expressions for the higher Euler-Selberg constants. We shall thus establish such formulas in the following section.

Remark 1.2. It would be interesting to ask, for instance, whether the Euler-Selberg constant (and its higher version) for Γ is algebraic or not? Is there any discrete subgroup Γ of which the Euler-Selberg constant is algebraic (modulo some invariant of the corresponding Riemann surface)? etc. Furthermore, numerical calculations of those constants seem also interesting.

Next, we consider an algebraic number field K. Recall the Dedekind zeta function;

$$\zeta_K(s) = \prod_{\mathfrak{p}}\left(1 - N(\mathfrak{p})^{-s}\right)^{-1} = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s},$$

where $\mathfrak{p}$ and $\mathfrak{a}$ respectively runs over prime ideals and integral ideals. Let

$$\zeta_K(s) = \frac{\gamma_{-1}(K)}{s-1} + \gamma_0(K) + \gamma_1(K)(s-1) + \gamma_2(K)(s-1)^2 + \cdots$$

be the Laurent expansion around $s=1$. We call

$$\gamma_0(K) = \lim_{s\to 1}\left(\zeta_K(s) - \frac{\gamma_{-1}(K)}{s-1}\right)$$

the Euler constant of K, and we may consider $\gamma_1(K), \gamma_2(K), \dots$ as higher Euler's constant of K. The residue $\gamma_{-1}(K)$ was calculated by Dedekind (1877) :

$$\gamma_{-1}(K) = \frac{2^{r_1}(2\pi)^{r_2} h(K) R(K)}{w(K)|D(K)|^{1/2}},$$

which is called the class number formula, where r_1 is the number of real infinite primes, r_2 is the number of complex infinite primes, $h(K)$ is the class number of K, $R(K)$ is the regulator of K, $w(K)$ is the number of roots of 1 in K, and $D(K)$ is the discriminant of K.

Theorem B. *Let K be an algebraic number field. Then we have*

(1) $\gamma_0(K) = \gamma_{-1}(K) \lim_{x \to \infty} \left(\log x - \sum_{N(\mathfrak{p})<x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p}) - 1} \right).$

(2) $$\sum_{\rho \in Z_K} \frac{1}{\rho} = \frac{\gamma_0(K)}{\gamma_{-1}(K)} - r_1 \left(\log 2 + \frac{\gamma}{2} \right) - r_2(\log 2 + \gamma) + \frac{1}{2} \log |D(K)| - \frac{[K:\mathbb{Q}]}{2} \log \pi + 1,$$

where ρ runs over the set Z_K of the essential zeroes of $\zeta_K(s)$.

The work in [KW1] has provided the one of motivations of the present study. In fact, both higher Euler's constants γ, γ_1 for $K = \mathbb{Q}$ and higher Euler-Selberg's constants $\gamma_\Gamma^{(0)}, \gamma_\Gamma^{(1)}$ are involved in the comparison formula of essential zeroes of the Riemann zeta and Selberg zeta functions established in [KW1].

2 Higher Euler-Selberg's constants

We prove first the following formulas of the Euler-Selberg constant $\gamma_\Gamma^{(0)}$ and the higher Euler-Selberg constants $\gamma_\Gamma^{(n)}$. After that we shall give formulas for the power sum over the essential zeroes of the Selberg zeta functions as the corollary of this result.

Theorem 1. *For any $n \geq 0$, we have*

$$\gamma_\Gamma^{(n)} = \frac{(-1)^n}{n!} \lim_{x \to \infty} \left\{ \sum_{\substack{\gamma \in \mathrm{Hyp}(\Gamma) \\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma) - 1} (\log N(\gamma))^n - \frac{(\log x)^{n+1}}{n+1} \right\}.$$

In particular, we have

$$\alpha_2(\Gamma) = \alpha_1(\Gamma) \lim_{x \to \infty} \left\{ \sum_{\substack{\gamma \in \mathrm{Hyp}(\Gamma) \\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma) - 1} - \log x \right\}.$$

It is clear that the assertion (1) in Theorem A is nothing but the statement of the case $n = 0$ in the theorem above.

Remark 2.1. Write the Laurent expansion of the Riemann zeta function as

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k.$$

Then one knows (Stieltjes, 1885; see Berndt [Ber], p.164) that the following formula holds.

$$\gamma_k = \frac{(-1)^k}{k\,!} \lim_{x\to\infty} \left\{ \sum_{n<x} \frac{(\log n)^k}{n} - \frac{(\log x)^{k+1}}{k+1} \right\}.$$

Thus the theorem above can be regarded as the Selberg zeta counterpart of the formulas of the higher Euler constants. However, if we look at them more precisely there are some differences between their respective denominators and exponents.

To show Theorem 1, we need some preparations.

2.1 Some lemmas

Lemma 2.1. *For any $n \geq 0$, we have*

$$\sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1} (\log N(\gamma))^n = \frac{(\log x)^{n+1}}{n+1} + A_n + O(x^{-\delta}) \quad \text{as } x \to \infty,$$

where A_n and $\delta > 0$ are some constants.
Proof. It is enough to consider

$$\sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)} (\log N(\gamma))^n,$$

since the difference

$$\sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{(N(\gamma)-1)N(\gamma)} (\log N(\gamma))^n$$

converges as $x \to \infty$. We may write the above as the Stieltjes integral:

$$\sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)} (\log N(\gamma))^n = \int_\tau^x \frac{(\log t)^n}{t} d\psi(t), \tag{2.1}$$

where we put

$$\psi(x) = \sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \log N(P_\gamma).$$

The following fact is known (see, for e.g. [Gan]),

$$\psi(x) = x + O(x^{1-\delta}) \quad \text{as } x \to \infty, \tag{2.2}$$

for some positive constant δ.

Put $\tau = \min_{\gamma \in \mathrm{Hyp}(\Gamma)} N(\gamma)$. Then we may calculate the right hand side of (2.1) as

$$\begin{aligned}
&\int_\tau^x \frac{(\log t)^n}{t} d\psi(t) \\
&= \frac{(\log x)^n}{x}\psi(x) - \int_\tau^x \frac{1}{t^2}\{n(\log t)^{n-1} - (\log t)^n\}\psi(t)dt \\
&= \frac{(\log x)^n}{x}(x + O(x^{1-\delta})) - \int_\tau^x \frac{1}{t^2}\{n(\log t)^{n-1} - (\log t)^n\}(t + O(t^{1-\delta}))dt \\
&= (\log x)^n - n\int_\tau^x \frac{(\log t)^{n-1}}{t}dt + \int_\tau^x \frac{(\log t)^n}{t}dt + C_n + O(x^{-\delta'}) \\
&= \frac{(\log x)^{n+1}}{n+1} + C'_n + O(x^{-\delta'}) \quad \text{as } x \to \infty
\end{aligned}$$

for some constants C_n, C'_n depending on n and $\delta' > 0$. Hence the proof immediately follows. ■

Lemma 2.2. *If* $|\operatorname{Im} s|$ *is sufficiently large and* $\operatorname{Re} s \geq \frac{1}{2} + \delta$ $(0 < \delta < 1)$, *then*

$$\left|\frac{Z'_\Gamma(s)}{Z_\Gamma(s)}\right| = O\Big(\delta^{-1}(\operatorname{Im} s)^{2\max(0,1+\delta-\operatorname{Re} s)}\Big).$$

Proof. For the proof, see [Hej2], Chap2, Prop 6.7. ■

We also recall the following well-known formula of the inverse Mellin transform.

Lemma 2..3. *If* $y > 0$ *and* $n \geq 0$, *then we have*

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} \frac{y^s}{s^{n+1}}ds = \begin{cases} \frac{1}{n!}(\log y)^n & (y > 1) \\ 0 & (0 < y \leq 1). \end{cases}$$

2.2 The case $n = 0$

We calculate the following limit of integral in two ways.

$$I(\gamma_\Gamma^{(0)}) := \frac{1}{2\pi i}\lim_{T\to\infty}\int_{2-iT}^{2+iT} \frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}ds.$$

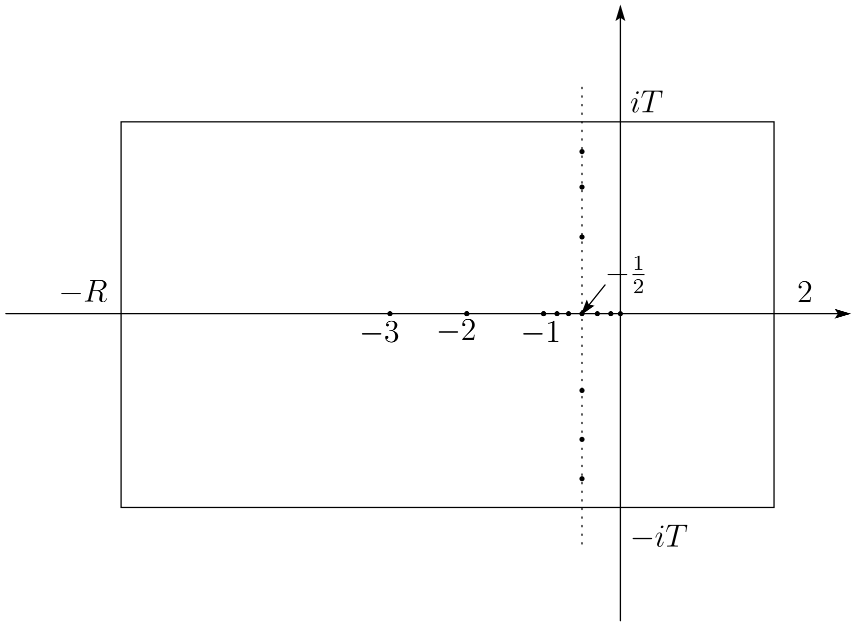
From the Euler product of $Z_\Gamma(s)$, using Lemma 2.3 we have

$$\begin{aligned}
I(\gamma_\Gamma^{(0)}) &= \sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1}\frac{1}{2\pi i}\lim_{T\to\infty}\int_{2-iT}^{2+iT} \frac{1}{s(s+1)}\Big(\frac{x}{N(\gamma)}\Big)^s ds \\
&= \sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1}\frac{1}{2\pi i}\lim_{T\to\infty}\int_{2-iT}^{2+iT} \Big(\frac{1}{s} - \frac{1}{s+1}\Big)\Big(\frac{x}{N(\gamma)}\Big)^s ds \\
&= \sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1}\Big(1 - \frac{N(\gamma)}{x}\Big).
\end{aligned}$$

Therefore, by the result of Lemma 2.1 and (2.2) it follows that

$$
\begin{aligned}
I(\gamma_\Gamma^{(0)}) &= (1-\frac{1}{x})\sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}}\frac{\log N(P_\gamma)}{N(\gamma)-1}-\frac{1}{x}\sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}}\log N(P_\gamma).\\
&=\sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}}\frac{\log N(P_\gamma)}{N(\gamma)-1}-1+o(1) \qquad \text{as } x\to\infty. \qquad (2.3)
\end{aligned}
$$

We now calculate the integral $I(\gamma_\Gamma^{(0)})$ in another way by the residue theorem. So we introduce the following contour $\tilde{C}_{R,T}$.



By (1.4), the poles and the residues of $Z_\Gamma'(s+1)/Z_\Gamma(s+1)$ are given as follows;

pole	residue
$s=\frac{1}{2}\pm ir_l(l>0)$	m_l (multiplicity of the eigenvalue λ_l)
$s=0$	1
$s=-1$	$2g-1$
$s=-k(k\geq 2)$	$2(g-1)(2k-1)$.

By the residue theorem, we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\tilde{C}_{R,T}}\frac{Z_\Gamma'(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}ds\\
&=\left\{\sum_{|r_l|<T}\Big(\operatorname*{Res}_{s=-\frac{1}{2}+ir_l}+\operatorname*{Res}_{s=-\frac{1}{2}-ir_l}\Big)+\sum_{k=2}^{[R]}\operatorname*{Res}_{s=-k}+\operatorname*{Res}_{s=-1}+\operatorname*{Res}_{s=0}\right\}\frac{Z_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}. \qquad (2.4)
\end{aligned}
$$

Now we calculate each term in the right hand side of (2.4).

Since $s = -\frac{1}{2} \pm ir_l$ are simple poles, it is immediate to see that

$$\operatorname*{Res}_{s=-\frac{1}{2}\pm ir_l} \frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} = m_l \frac{x^{-\frac{1}{2}\pm ir_l}}{-\frac{1}{4} - r_l^2}.$$

Thus we have

$$\sum_{|r_l|<T} \Big(\operatorname*{Res}_{s=-\frac{1}{2}+ir_l} + \operatorname*{Res}_{s=-\frac{1}{2}-ir_l} \Big) \frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} = -\sum_{|r_l|<T} m_l \frac{x^{-\frac{1}{2}} \cos(r_l \log x)}{\frac{1}{4} + r_l^2}. \quad (2.5)$$

The pole at $s = -k (k \geq 2)$ is also simple, so that

$$\begin{aligned} \operatorname*{Res}_{s=-k} \frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} &= 2(g-1) \frac{2k-1}{k(k-1)} x^{-k} \\ &= 2(g-1) \Big(\frac{1}{k-1} + \frac{1}{k} \Big) x^{-k}. \end{aligned}$$

By the elementary fact

$$\sum_{k=1}^{\infty} \frac{x^{-k}}{k} = -\log(1 - \frac{1}{x}),$$

we have

$$\begin{aligned} &\sum_{k=2}^{[R]} \operatorname*{Res}_{s=-k} \frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} \\ &\xrightarrow{R\to\infty} 2(g-1)\{-x^{-1} - \log(1 - x^{-1}) - x^{-1} \log(1 - x^{-1})\} = o(1) \quad \text{as } x \to \infty. \quad (2.6) \end{aligned}$$

Since $s = -1$ and 0 are double poles, consider the Laurent expansion. First, around $s = -1$ we observe that

$$\begin{aligned} &\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} \\ &\quad = \Big((2g-1)(s+1)^{-1} + a_0 + a_1(s+1) + \dots \Big) \\ &\quad \times \Big(1 + (s+1) + \dots \Big) \times (s+1)^{-1} \times x^{-1} \Big(1 + (\log x)(s+1) + \dots \Big) \\ &\quad = (2g-1)x^{-1}(s+1)^{-2} + x^{-1} \Big((2g-1)(\log x + 1) + a_0 \Big) (s+1)^{-1} + \dots, \end{aligned}$$

whence one has

$$\operatorname*{Res}_{s=-1} \frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} = x^{-1} \Big((2g-1)(\log x + 1) + a_0 \Big) = o(1) \quad \text{as } x \to \infty. \quad (2.7)$$

Similarly, around $s = 0$ one may expand

$$\begin{aligned} &\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} \\ &\quad = \Big(s^{-1} + \gamma_\Gamma + \gamma_\Gamma^{(1)} s + \dots \Big) \times s^{-1} \times \Big(1 - s + \dots \Big) \times \Big(1 + (\log x)s + \dots \Big) \\ &\quad = s^{-2} + (\log x + \gamma_\Gamma - 1)s^{-1} + \dots. \end{aligned}$$

Hence we have

$$\operatorname*{Res}_{s=0} \frac{Z_\Gamma'(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} = \log x + \gamma_\Gamma - 1. \tag{2.8}$$

By substituting these above into (2.4), we conclude that

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\tilde{C}_{R,T}} \frac{Z_\Gamma'(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} ds \\ &= \log x + \gamma_\Gamma - 1 - \sum_{|r_l|<T} m_l \frac{x^{-\frac{1}{2}} \cos(r_l \log x)}{\frac{1}{4} + r_l^2} + o(1) \quad \text{as } x \to \infty. \end{aligned} \tag{2.9}$$

Next we estimate the integration along $\tilde{C}_{R,T}$ excluding the line $\operatorname{Re} s = 2$. In order to do this, we divide this integral into the following eight parts.

$$\begin{aligned} &\int_{\tilde{C}_{R,T}} \frac{Z_\Gamma'(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} ds \\ &= \int_{2-iT}^{2+iT} + \int_{2+iT}^{-\frac{1}{2}+\varepsilon+iT} + \int_{-\frac{1}{2}+\varepsilon+iT}^{-\frac{1}{2}-\varepsilon+iT} + \int_{-\frac{1}{2}-\varepsilon+iT}^{-R+iT} \\ &+ \int_{-R+iT}^{-R-iT} + \int_{-R-iT}^{-\frac{1}{2}-\varepsilon-iT} + \int_{-\frac{1}{2}-\varepsilon-iT}^{-\frac{1}{2}+\varepsilon-iT} + \int_{-\frac{1}{2}+\varepsilon-iT}^{2-iT} \frac{Z_\Gamma'(s+1)}{Z_\Gamma(s+1)} \frac{x^s}{s(s+1)} ds, \end{aligned}$$

where ε is taken such as $0 < \varepsilon < \frac{1}{2}$.

We denote by $I_1, \ldots, I_8$ respectively the integrals of the right hand side of the above. Note that $I(\gamma_\Gamma^{(0)}) = \frac{1}{2\pi i} \lim_{T\to\infty} I_1$. Our task is then to estimate $I_2, I_3, \ldots, I_8$.

[**Estimate of** I_5:] By the functional equation (1.8) of the logarithmic derivative of $Z_\Gamma(s)$, we have

$$\begin{aligned} I_5 &= \int_{-R+iT}^{-R-iT} \Big\{ -\frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)} + \mu(\Gamma\backslash H)(s+\frac{1}{2}) \cot \pi s \Big\} \frac{x^s}{s(s+1)} ds \\ &= -\int_{-R+iT}^{-R-iT} \frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)} \frac{x^s}{s(s+1)} ds + \mu(\Gamma\backslash H) \int_{-R+iT}^{-R-iT} (s+\frac{1}{2}) \cot \pi s \frac{x^s}{s(s+1)} ds. \end{aligned} \tag{2.10}$$

The first term of (2.10) is estimated as follows.

$$\left| \int_{-R+iT}^{-R-iT} \frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)} \frac{x^s}{s(s+1)} ds \right| \le \int_{-T}^{T} \frac{Z_\Gamma'(R)}{Z_\Gamma(R)} \frac{x^{-R}}{t^2+R^2} dt.$$

Since $\lim_{T\to\infty} \int_{-T}^{T} \frac{1}{t^2+R^2} dt$ exists, we have

$$\left| \int_{-R+iT}^{-R-iT} \frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)} \frac{x^s}{s(s+1)} ds \right| \xrightarrow{T,R\to\infty} 0.$$

It is clear that the second term of (2.10) can be written as

$$\int_{-R+iT}^{-R-iT} (s+\frac{1}{2}) \cot \pi s \frac{x^s}{s(s+1)} ds = \frac{1}{2} \int_{-R+iT}^{-R-iT} \Big(\frac{1}{s} + \frac{1}{s+1}\Big) x^s \cot \pi s ds.$$

It is also easy to see that

$$\int_{-R+iT}^{-R-iT} \frac{x^s}{s} \cot \pi s ds \overset{T,R\to\infty}{\longrightarrow} 0,$$

and

$$\int_{-R+iT}^{-R-iT} \frac{x^s}{s+1} \cot \pi s ds \overset{T,R\to\infty}{\longrightarrow} 0.$$

Hence we have

$$I_5 \overset{T,R\to\infty}{\longrightarrow} 0. \tag{2.11}$$

[**Estimate of** I_2 **and** I_8:] By applying Lemma 2.2 for a fixed $\delta(<\varepsilon)$, we have for I_2

$$|I_2| \le \int_2^{-\frac{1}{2}+\varepsilon} O\Big(\delta^{-1}T^{2\max(0,\delta-t)}\Big)\frac{x^t}{T^2}dt = O\Big(T^{-1}\Big). \tag{2.12}$$

Similarly

$$I_8 \overset{T\to\infty}{\longrightarrow} 0. \tag{2.13}$$

[**Estimate of** I_4 **and** I_6:] By the functional equation (1.8) again, we have

$$I_4 = \int_{-\frac{1}{2}-\varepsilon+iT}^{-R+iT} \Big\{-\frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)} + \mu(\Gamma\backslash H)(s+\frac{1}{2})\cot \pi s\Big\}\frac{x^s}{s(s+1)}ds.$$

It follows from Lemma 2.2 that

$$\left|\int_{-\frac{1}{2}-\varepsilon+iT}^{-R+iT} \frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)}\frac{x^s}{s(s+1)}ds\right| \le \int_{-\frac{1}{2}-\varepsilon}^{-R} O(T)\frac{x^t}{T^2}dt.$$

Since the integral $\int_{-\frac{1}{2}-\varepsilon}^{-R} x^t dt$ converges as $R\to\infty$, it is clear that

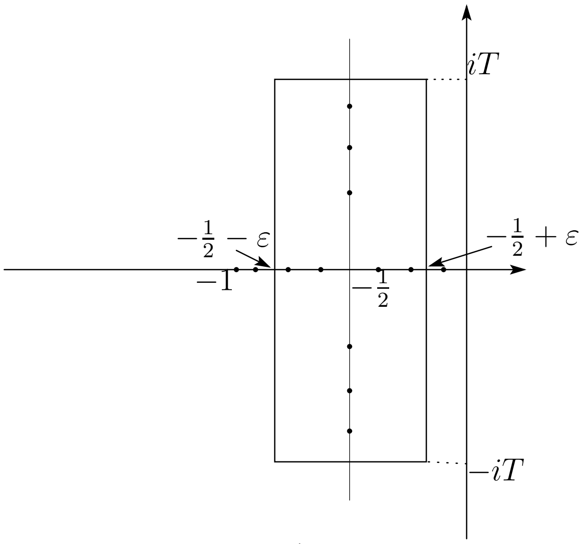
$$\left|\int_{-\frac{1}{2}-\varepsilon+iT}^{-R+iT} \frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)}\frac{x^s}{s(s+1)}ds\right| \overset{R,T\to\infty}{\longrightarrow} 0.$$

Also since

$$\left|\int_{-\frac{1}{2}-\varepsilon+iT}^{-R+iT} (s+\frac{1}{2})\cot \pi s\frac{x^s}{s(s+1)}ds\right| \le \int_{-\frac{1}{2}-\varepsilon}^{-R} o(1)\frac{x^t}{T}dt \overset{R,T\to\infty}{\longrightarrow} 0.$$

one obtains $I_4 \overset{R,T\to\infty}{\longrightarrow} 0$. Quite similarly we have $I_6 \overset{R,T\to\infty}{\longrightarrow} 0$.

[**Estimate of** I_3 **and** I_7:] To estimate I_3 and I_7 we consider the following contour C_T^{ε}.



Inside of the contour C_T^{ε}, the poles of $\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}$ are located at $s=-\frac{1}{2}\pm ir_l$, where r_l is either real with $|r_l|<T$, or pure imaginary with $|r_l|<\varepsilon$. Hence by the residue theorem, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_T^{\varepsilon}}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}ds &= \sum_{\substack{r_l\in\mathbb{R},|r_l|<T,\\ \text{or } r_l\in i\mathbb{R},|r_l|<\varepsilon}}\Big(\operatorname*{Res}_{s=-\frac{1}{2}+ir_l}+\operatorname*{Res}_{s=-\frac{1}{2}-ir_l}\Big)\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}\\ &= -\sum_{\substack{r_l\in\mathbb{R},|r_l|<T,\\ \text{or } r_l\in i\mathbb{R},|r_l|<\varepsilon}} m_l\frac{x^{-\frac{1}{2}}\cos(r_l\log x)}{\frac{1}{4}+r_l^2}.\end{aligned}$$

Since the number of $r_l\in i\mathbb{R}$ is finite, the right hand side of the above equation turns to be

$$-\sum_{|r_l|<T} m_l\frac{x^{-\frac{1}{2}}\cos(r_l\log x)}{\frac{1}{4}+r_l^2}+o(1)\quad\text{as } x\to\infty.$$

Moreover, since the left hand side is equal to

$$I_3+I_7+\int_{-\frac{1}{2}+\varepsilon-iT}^{-\frac{1}{2}+\varepsilon+iT}+\int_{-\frac{1}{2}-\varepsilon+iT}^{-\frac{1}{2}-\varepsilon-iT}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}ds,$$

we find

$$\begin{aligned}&-\sum_{|r_l|<T} m_l\frac{x^{-\frac{1}{2}}\cos(r_l\log x)}{\frac{1}{4}+r_l^2}-\frac{1}{2\pi i}(I_3+I_7)\\ &=\frac{1}{2\pi i}\left\{\int_{-\frac{1}{2}+\varepsilon-iT}^{-\frac{1}{2}+\varepsilon+iT}+\int_{-\frac{1}{2}-\varepsilon+iT}^{-\frac{1}{2}-\varepsilon-iT}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}ds\right\}+o(1)\quad\text{as } x\to\infty. \qquad (2.14)\end{aligned}$$

Now we call respectively by I_9, I_{10} the integrals of the right hand side of (2.14). It is then enough to estimate I_9 and I_{10}. Take sufficiently large number $T'(<T)$ and consider I_9 as

$$I_9=\int_{-\frac{1}{2}+\varepsilon+T'}^{-\frac{1}{2}+\varepsilon+T}+\int_{-\frac{1}{2}+\varepsilon-T'}^{-\frac{1}{2}+\varepsilon+T'}+\int_{-\frac{1}{2}+\varepsilon-T}^{-\frac{1}{2}+\varepsilon-'T}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s(s+1)}ds.$$

By Lemma 2.1, the first integral of the expression of I_9 above is bounded by

$$\int_{T'}^{T} O(t^{1-\varepsilon})\frac{x^{-\frac{1}{2}+\varepsilon}}{t^2+\varepsilon^2}dt \overset{T\to\infty}{\longrightarrow} O(x^{-\frac{1}{2}+\varepsilon}) \quad \text{as } x\to\infty.$$

The exactly same estimate holds for the third integral. Also it is clear that the second integral is $O(x^{-\frac{1}{2}+\varepsilon})$. Hence we have

$$I_9 \overset{T\to\infty}{\longrightarrow} O(x^{-\frac{1}{2}+\varepsilon}) \quad \text{as } x\to\infty. \tag{2.15}$$

Next, by (1.8) we see that

$$I_{10} = \int_{-\frac{1}{2}-\varepsilon+iT}^{-\frac{1}{2}-\varepsilon-iT} \left\{-\frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)} + \mu(\Gamma\backslash H)(s+\frac{1}{2})\cot\pi s\right\}\frac{x^s}{s(s+1)}ds.$$

A similar argument as we made above yields

$$\left|\int_{-\frac{1}{2}-\varepsilon+iT}^{-\frac{1}{2}-\varepsilon-iT}\frac{Z_\Gamma'(-s)}{Z_\Gamma(-s)}\frac{x^s}{s(s+1)}ds\right| \overset{T\to\infty}{\longrightarrow} O(x^{-\frac{1}{2}-\varepsilon}) \quad \text{as } x\to\infty.$$

Since it is also obvious to see that

$$\int_{-\frac{1}{2}-\varepsilon+iT}^{-\frac{1}{2}-\varepsilon-iT}\cot\pi s\frac{x^s}{s}ds \overset{T\to\infty}{\longrightarrow} O(x^{-\frac{1}{2}-\varepsilon}) \quad \text{as } x\to\infty,$$

we have

$$I_{10} \overset{T\to\infty}{\longrightarrow} O(x^{-\frac{1}{2}+\varepsilon}) \quad \text{as } x\to\infty. \tag{2.16}$$

By these estimates for I_9, I_{10}, we have

$$\left|\sum_{|r_l|<T} m_l \frac{x^{-\frac{1}{2}}\cos(r_l\log x)}{\frac{1}{4}+r_l^2} + \frac{1}{2\pi i}(I_3+I_7)\right| \overset{T\to\infty}{\longrightarrow} o(1) \quad \text{as } x\to\infty. \tag{2.17}$$

Combining the results we obtained so far, we conclude that

$$\sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1} = \log x + \gamma_\Gamma + o(1) \quad \text{as } x\to\infty.$$

This completes the proof of the case $n=0$. ■

2.3 The case $n \geq 1$

To prove the case for $n\geq 1$, we consider the integral.

$$J(\gamma_\Gamma^{(n)}) =: \frac{1}{2\pi i}\lim_{T\to\infty}\int_{2-iT}^{2+iT}\frac{Z_\Gamma'(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s^{n+1}}ds. \tag{2.18}$$

Similar to the case of $n = 0$, using the definition of $Z_\Gamma(s)$, one calculates

$$\begin{aligned}
J(\gamma_\Gamma^{(n)}) &= \sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1}\frac{1}{2\pi i}\lim_{T\to\infty}\int_{2-iT}^{2+iT}\frac{1}{s^{n+1}}(\frac{x}{N(\gamma)})^s ds.\\
&= \sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1}\frac{1}{n!}(\log\frac{x}{N(\gamma)})^n \qquad \text{(by Lemma 2.3)}\\
&= \sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}} \frac{\log N(P_\gamma)}{N(\gamma)-1}\frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}(-1)^l(\log x)^{n-l}(\log N(\gamma))^l\\
&= \sum_{l=0}^{n}\frac{(-1)^l}{l!(n-l)!}(\log x)^{n-l}\sum_{\substack{\gamma\in \mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}}\frac{\log N(P_\gamma)}{N(\gamma)-1}(\log N(\gamma))^l\\
&= \sum_{l=0}^{n}\frac{(-1)^l}{l!(n-l)!}(\log x)^{n-l}\Big\{\frac{1}{l+1}(\log x)^{l+1}+A_l+O(x^{-\delta})\Big\} \qquad \text{(by Lemma 2.1)}\\
&= \frac{(\log x)^{n+1}}{(n+1)!}\sum_{l=0}^{n}\binom{n+1}{l+1}(-1)^l+\sum_{l=0}^{n}\frac{(-1)^l}{l!(n-l)!}(\log x)^{n-l}A_l+O(x^{-\delta})\\
&= \frac{(\log x)^{n+1}}{(n+1)!}+\sum_{l=0}^{n}\frac{(\log x)^{n-l}}{(n-l)!}\frac{(-1)^l}{l!}A_l+O(x^{-\delta}) \quad \text{as } x\to\infty. \qquad (2.19)
\end{aligned}$$

On the other hand, we consider the same contour $\tilde{C}_{R,T}$ as we used before. Then the residue theorem asserts

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\tilde{C}_{R,T}}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s^{n+1}}ds \qquad (2.20)\\
&=\Bigg\{\sum_{|r_l|<T}(\operatorname*{Res}_{s=-\frac12+ir_l}+\operatorname*{Res}_{s=-\frac12-ir_l})+\sum_{k=2}^{[R]}\operatorname*{Res}_{s=-k}+\operatorname*{Res}_{s=-1}+\operatorname*{Res}_{s=0}\Bigg\}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s^{n+1}}.
\end{aligned}$$

It is not hard to confirm the following three evaluations.

$$\sum_{|r_l|<T}(\operatorname*{Res}_{s=-\frac12+ir_l}+\operatorname*{Res}_{s=-\frac12-ir_l})\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s^{n+1}}=\sum_{|r_l|<T}m_l\Bigg\{\frac{x^{-\frac12+ir_l}}{(-\frac12+ir_l)^{n+1}}+\frac{x^{-\frac12-ir_l}}{(-\frac12-ir_l)^{n+1}}\Bigg\}. \qquad (2.21)$$

$$\sum_{k=2}^{[R]}\operatorname*{Res}_{s=-k}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s^{n+1}}=\sum_{k=2}^{[R]}2(g-1)(2k-1)\frac{x^{-k}}{(-k)^{n+1}}=o(1) \quad \text{as } x\to\infty. \qquad (2.22)$$

$$\operatorname*{Res}_{s=-1}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s^{n+1}}=(2g-1)(-1)^{n+1}x^{-1}=o(1) \quad \text{as } x\to\infty. \qquad (2.23)$$

Moreover, since one has

$$\begin{aligned}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s^{n+1}} &= \Big(\sum_{l=0}^{\infty}\gamma_\Gamma^{(l-1)}s^{l-1}\Big)\Big(\sum_{m=0}^{\infty}\frac{(\log x)^m}{m!}s^m\Big)s^{-n-1}\\ &= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\gamma_\Gamma^{(l-1)}\frac{(\log x)^m}{m!}s^{m+l-n-2}\\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\gamma_\Gamma^{(l-1)}\frac{(\log x)^{k-l}}{(k-l)!}s^{k-n-2}\end{aligned}$$

around $s=0$, it follows that

$$\begin{aligned}\operatorname*{Res}_{s=0}\frac{Z'_\Gamma(s+1)}{Z_\Gamma(s+1)}\frac{x^s}{s^{n+1}} &= \sum_{l=0}^{n+1}\gamma_\Gamma^{(l-1)}\frac{(\log x)^{n+1-l}}{(n+1-l)!}\\ &= \frac{(\log x)^{n+1}}{(n+1)!}+\sum_{l=0}^{n}\gamma_\Gamma^{(l)}\frac{(\log x)^{n-l}}{(n-l)!}. \end{aligned} \tag{2.24}$$

Using the estimates above we see that

$$\begin{aligned}J(\gamma_\Gamma^{(n)}) &= \sum_{|r_l|<T} m_l\left\{\frac{x^{-\frac12+ir_l}}{(-\frac12+ir_l)^{n+1}}+\frac{x^{-\frac12-ir_l}}{(-\frac12-ir_l)^{n+1}}\right\}\\ &\quad+\frac{(\log x)^{n+1}}{(n+1)!}+\sum_{l=0}^{n}\gamma_\Gamma^{(l)}\frac{(\log x)^{n-l}}{(n-l)!}+o(1)\quad \text{as } x\to\infty. \end{aligned}\tag{2.25}$$

Similar estimates work for $J(\gamma_\Gamma^{(n)})$ as we done in the case of $n=0$. Thus, when $x\to\infty$ we have

$$\frac{(\log x)^{n+1}}{(n+1)!}+\sum_{l=0}^{n}\frac{(\log x)^{n-l}}{(n-l)!}\frac{(-1)^l}{l!}A_l=\frac{(\log x)^{n+1}}{(n+1)!}+\sum_{l=0}^{n}\gamma_\Gamma^{(l)}\frac{(\log x)^{n-l}}{(n-l)!}+o(1)$$

This means clearly that $\gamma_\Gamma^{(n)}=\frac{(-1)^n}{n!}A_n$. We therefore obtain from Lemma 2.1 that

$$\gamma_\Gamma^{(n)}=\frac{(-1)^n}{n!}A_n=\frac{(-1)^n}{n!}\lim_{x\to\infty}\left\{\sum_{\substack{\gamma\in\mathrm{Hyp}(\Gamma)\\ N(\gamma)<x}}\frac{\log N(P_\gamma)}{N(\gamma)-1}(\log N(\gamma))^n-\frac{(\log x)^{n+1}}{n+1}\right\}. \tag{2.26}$$

This proves the case $n\geq 1$ and Theorem 1 now follows. ■

2.4 Powers of essential zeroes of the Selberg zeta function

We prove the statement (2) in Theorem B. In fact, we establish the formulas which express the sum of the integral powers of essential zeroes of the Selberg zeta function.

Define the spectral zeta function $\zeta_\Delta(s)$ by

$$\zeta_\Delta(s)=\sum_{n=1}^{\infty}\lambda_n^{-s}=\sum_{n=1}^{\infty}(\frac14+r_n^2)^{-s}.$$

From the expression (1.4) the
following Laurent expansion around $s=1$ ([Ste]) follows immediately.

$$\frac{1}{2s-1}\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}=\frac{1}{s-1}+\sum_{n=0}^{\infty}a_n(s-1)^n,$$

where the coefficients a_n are given by

$$a_0=\lim_{s\to1}\Big\{\frac{1}{2s-1}\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}-\frac{1}{s-1}\Big\},$$
$$a_n=(-1)^{n+1}\Big\{1+2(g-1)\zeta(n+1)+\sum_{l=0}^{[\frac{n}{2}]}(-1)^{l+1}\binom{n-l}{l}\zeta_\Delta(n+1-l)\Big\}.$$

By Theorem 2.1 one easily has

$$\gamma_\Gamma^{(n)}=2a_{n-1}+a_n, \tag{2.27}$$

where we put $a_{-1}=1$ for convenience. Hence it follows that

$$\gamma_\Gamma^{(0)}=2+a_0.$$

If we put $n=1$ in (2.27) then since $\gamma_\Gamma^{(1)}=2a_0+a_1$ we have

$$-A_1=2(A_0-2)+\{1+2(g-1)\zeta(2)-\zeta_\Delta(2)\},$$

whence

$$\zeta_\Delta(2)=A_1+2A_0+2(g-1)\zeta(2)-3.$$

Suppose $n\geq 2$.. Then

$$\frac{(-1)^n}{n!}A_n=2(-1)^n\Big\{1+2(g-1)\zeta(n)+\sum_{l=0}^{[\frac{n-1}{2}]}(-1)^{l+1}\binom{n-1-l}{l}\zeta_\Delta(n-l)\Big\}$$
$$+(-1)^{n+1}\Big\{1+2(g-1)\zeta(n+1)+\sum_{l=0}^{[\frac{n}{2}]}(-1)^{l+1}\binom{n-l}{l}\zeta_\Delta(n+1-l)\Big\}.$$

Namely we have

$$\frac{A_n}{n!}=1+2(g-1)\Big\{2\zeta(n)-\zeta(n+1)\Big\}+2\sum_{l=0}^{[\frac{n-1}{2}]}(-1)^{l+1}\binom{n-1-l}{l}\zeta_\Delta(n-l)$$
$$-\sum_{l=0}^{[\frac{n}{2}]-1}(-1)^{l+1}\binom{n-1-l}{l+1}\zeta_\Delta(n-l)+\zeta_\Delta(n+1).$$

Thus we obtain the following result.

Corollary. The values $\zeta_\Delta(n)\,(n \geq 2)$ is determined successively by the following recurrence formulas.

$$\begin{aligned}
\zeta_\Delta(2) &= A_1 + 2A_0 + 2(g-1)\zeta(2) - 3 \\
&= 2\gamma_\Gamma^{(0)} - \gamma_\Gamma^{(1)} + (g-1)\frac{\pi^2}{3} - 3, \\
\zeta_\Delta(n+1) &= \sum_{l=0}^{[\frac{n}{2}]-1} (-1)^l \left\{ \binom{n-1-l}{l+1} + 2\binom{n-1-l}{l} \right\} \zeta_\Delta(n-l) \\
&\quad + \{(-1)^{n+1}+1\}(-1)^{[\frac{n+1}{2}]} + \zeta_\Delta([\frac{n+1}{2}]) \\
&\quad + \frac{A_n}{n!} - 2(g-1)\Big\{2\zeta(n) - \zeta(n+1)\Big\} - 1.
\end{aligned}$$

3 Euler's constants of algebraic number fields

We give the proof of Theorem B concerning the Euler constants of algebraic number fields. The way of proving (1) is quite similar to the one for the case of $n = 0$ in Theorem A. Thus we shall skip some detail of the estimates.

Proof of Theorem B (1). Since

$$-\frac{\zeta_K{}'(s)}{\zeta_K(s)} = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{ms}}$$

where $\mathfrak{p}$ runs over prime ideals, by Lemma 2.3 with $n = 0$ we see that

$$\begin{aligned}
-\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta_K{}'(s+1)}{\zeta_K(s+1)} x^s s^{-1} ds &= \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^m} \cdot \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \left(\frac{x}{N(\mathfrak{p})^m}\right)^s s^{-1} ds \\
&= \sum_{N(\mathfrak{p})^m < x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^m} + o(1) \\
&= \sum_{N(\mathfrak{p}) < x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p}) - 1} + o(1)
\end{aligned}$$

when $x \to \infty$. We refer to Narkiewicz [Nar] for basic analytic properties of the Dedekind zeta functions.

Take a path $C_{R,T} : 2 - iT \to -R - iT \to -R + iT \to 2 + iT$ for suitable large numbers T and R. (Note that $C_{R,T} = (-\tilde{C}_{R,T}) \cup [2 - iT, 2 + iT]$ in the previously defined contour $\tilde{C}_{R,T}$). Then using estimates for the Dedekind zeta function, as we did in the Section 2, we have

$$\begin{aligned}
&\lim_{T\to\infty} \left\{ -\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{\zeta_K{}'(s+1)}{\zeta_K(s+1)} x^s s^{-1} ds \right\} \\
= \ &\lim_{T\to\infty}\lim_{R\to\infty} \left(\int_{2-iT}^{2+iT} - \int_{C_{R,T}} \right) \left(-\frac{1}{2\pi i} \cdot \frac{\zeta_K{}'(s+1)}{\zeta_K(s+1)} x^s s^{-1} \right) ds \\
&\quad + \lim_{T\to\infty}\lim_{R\to\infty} \left\{ -\frac{1}{2\pi i}\int_{C_{R,T}} \frac{\zeta_K{}'(s+1)}{\zeta_K(s+1)} x^s s^{-1} ds \right\} \\
= \ &\log x - \frac{\gamma_0(K)}{\gamma_{-1}(K)} + o(1) \quad \text{as } x \to \infty
\end{aligned}$$

since

$$\frac{{\zeta_K}'(s)}{\zeta_K(s)} = -\frac{1}{s-1} + \frac{\gamma_0(K)}{\gamma_{-1}(K)} + \cdots \tag{3.1}$$

around $s = 1$. Hence it is immediate to see

$$\sum_{N(\mathfrak{p})<x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})-1} = \log x - \frac{\gamma_0(K)}{\gamma_{-1}(K)} + o(1) \quad \text{as } x \to \infty.$$

Hence it follows that

$$\gamma_0(K) = \gamma_{-1}(K) \lim_{x\to\infty} \left(\log x - \sum_{N(\mathfrak{p})<x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})-1} \right).$$

Proof of (2). Put

$$\hat{\zeta}_K(s) = s(1-s) \left(\frac{\sqrt{|D(K)|}}{2^{r_2}\pi^{[K:\mathbb{Q}]/2}} \right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s). \tag{3.2}$$

Then, it is known that the following functional equation holds.

$$\hat{\zeta}_K(s) = \hat{\zeta}_K(1-s). \tag{3.3}$$

Since $\hat{\zeta}_K(s)$ is of order 1, one may write

$$\hat{\zeta}_K(s) = e^{as+b} \prod_{\rho\in Z_K} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \tag{3.4}$$

with some complex numbers a and b.

Taking the logarithmic derivative of right-hand sides of (3.2) and (3.4) respectively, we have

$$a + \sum_{\rho\in Z_K} \left(\frac{-\frac{1}{\rho}}{1-\frac{s}{\rho}} + \frac{1}{\rho} \right) = \frac{1}{s} + \frac{-1}{1-s} + \log \frac{\sqrt{|D(K)|}}{2^{r_2}\pi^{[K:\mathbb{Q}]/2}} + r_1 \frac{\Gamma'\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)} + r_2 \frac{\Gamma'(s)}{\Gamma(s)} + \frac{{\zeta_K}'(s)}{\zeta_K(s)}. \tag{3.5}$$

By the functional equation (3.3), one sees that the left-hand side of (3.5) is equal to also

$$-a - \sum_{\rho\in Z_K} \left(\frac{-\frac{1}{\rho}}{1-\frac{1-s}{\rho}} + \frac{1}{\rho} \right).$$

It is hence clear that

$$-2a = \sum_{\rho\in Z_K} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + \sum_{\rho\in Z_K} \left(\frac{1}{1-s-\rho} + \frac{1}{\rho} \right).$$

Putting $s = 1$ yields

$$a = -\frac{1}{2} \sum_{\rho\in Z_K} \left(\frac{1}{\rho} + \frac{1}{1-\rho} \right).$$

Hence it follows from (3.5) that

$$-\frac{1}{2}\sum_{\rho\in Z_K}\left(\frac{1}{\rho}+\frac{1}{1-\rho}\right)+\sum_{\rho\in Z_K}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)=\frac{1}{s}+\frac{1}{s-1}+\frac{1}{2}\log|D(K)|-$$
$$\frac{[K:\mathbb{Q}]}{2}\log\pi+r_1\frac{\Gamma'\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)}+r_2\left\{\frac{\Gamma'(s)}{\Gamma(s)}-\log 2\right\}+\frac{\zeta_K{}'(s)}{\zeta_K(s)}.$$

Therefore if we let $s\to 1$ and use the expression (3.1) we obtain

$$\begin{aligned}&\frac{1}{2}\sum_{\rho\in Z_K}\left(\frac{1}{1-\rho}+\frac{1}{\rho}\right)\\ =\ &1+\frac{1}{2}\log|D(K)|-\frac{[K:\mathbb{Q}]}{2}\log\pi+r_1\frac{\Gamma'\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)}+r_2\left\{\frac{\Gamma'(1)}{\Gamma(1)}-\log 2\right\}+\frac{\gamma_0(K)}{\gamma_{-1}(K)}.\end{aligned}$$

Since one knows that $\Gamma'\left(\frac{1}{2}\right)/\Gamma\left(\frac{1}{2}\right)=-\gamma-2\log 2$ and $\Gamma'(1)/\Gamma(1)=-\gamma$, the following equation follows immediately.

$$\sum_{\rho\in Z_K}\frac{1}{\rho}=\frac{\gamma_0(K)}{\gamma_{-1}(K)}-r_1\left(\log 2+\frac{\gamma}{2}\right)-r_2(\log 2+\gamma)+\frac{1}{2}\log|D(K)|-\frac{[K:\mathbb{Q}]}{2}\log\pi+1.$$

This completes the proof of the theorem.

Example 3.1. Consider the case $K=\mathbb{Q}$. Then, since $n=1$, $r_1=1$, $r_2=0$, $D(\mathbb{Q})=1$, $\gamma_{-1}(\mathbb{Q})=1$ and $\gamma_0(\mathbb{Q})=\gamma$, we recover the Riemann's explicit formula (1.1) as follows.

$$\begin{aligned}\sum_{\rho\in Z_{\mathbb{Q}}}\frac{1}{\rho}&=\gamma-\left(\log 2+\frac{\gamma}{2}\right)-\frac{1}{2}\log\pi+1\\ &=1+\frac{\gamma}{2}-\frac{1}{2}\log\pi-\log 2.\end{aligned}$$

Remark 3.2. We remark here that the following another expression of $\gamma_0(K)$ is known.

$$\gamma_0(K)=\lim_{x\to\infty}\left(\sum_{N(\mathfrak{a})<x}\frac{1}{N(\mathfrak{a})}-\gamma_{-1}(K)\log x\right)\qquad\text{(Landau, 1902).}$$

4 Higher Euler's constants and powers of essential zeroes

In this section we give certain formulas of powers of essential zeros of the Riemann zeta function. It is actually considered as the higher power analogue of Riemann's explicit formula (1.1).

Theorem 3. *Let*

$$\zeta(s)=\frac{1}{s-1}+\gamma+\gamma_1(s-1)+\gamma_2(s-1)^2+\cdots$$

be the Laurent expansion around $s=1$. Then the following relations hold:

(1) $2\gamma_1 - \gamma^2 = 1 - \dfrac{\pi^2}{8} - \displaystyle\sum_{\rho\in Z} \frac{1}{\rho^2}$.

(2) $2\gamma_1 - \gamma^2 = \displaystyle\lim_{x\to\infty} \left\{ \frac{1}{2}(\log x)^2 - \gamma \log x - \sum_{n<x} \frac{\Lambda(n)}{n} \log \frac{x}{n} \right\}$.

(3) $3\gamma_2 - 3\gamma_1\gamma + \gamma^3 = \dfrac{7}{8}\zeta(3) - 1 + \displaystyle\sum_{\rho\in Z} \frac{1}{\rho^3}$.

(4) $$3\gamma_2 - 3\gamma_1\gamma + \gamma^3 = \lim_{x\to\infty} \left\{ \frac{1}{6}(\log x)^3 - \frac{1}{2}\gamma(\log x)^2 - (2\gamma_1 - \gamma^2)\log x \right. \\ \left. - \frac{1}{2}\sum_{n<x} \frac{\Lambda(n)}{n}\left(\log \frac{x}{n}\right)^2 \right\} .$$

Before starting the proof, we first note that the Laurent expansion.

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + (2\gamma_1 - \gamma^2)(s-1) + (3\gamma_2 - 3\gamma_1\gamma + \gamma^3)(s-1)^2 + \cdots . \quad (4.1)$$

For convenience, set

$$b_1 = 2\gamma_1 - \gamma^2, \qquad b_2 = 3\gamma_2 - 3\gamma_1\gamma + \gamma^3.$$

Proof of (1). Put

$$J := \lim_{T\to\infty} \left\{ -\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{\zeta'(s+1)}{\zeta(s+1)} s^{-2} ds \right\} . \quad (4.2)$$

Similar to the discussion we made in the Section 2, we evaluate J by changing the path. In fact, we show first that $J = 0$ as follows. Consider the path given by

$$C_T : 2 - iT \to T - iT \to T + iT \to 2 + iT.$$

We may choose T large enough. By the residue theorem it is obvious to see that

$$\left(\int_{C_T} - \int_{2-iT}^{2+iT} \right) \left(-\frac{1}{2\pi i} \cdot \frac{\zeta'(s+1)}{\zeta(s+1)} s^{-2} \right) ds = 0.$$

Therefore, since

$$\left| \frac{1}{2\pi i} \int_{C_T} \frac{\zeta'(s+1)}{\zeta(s+1)} s^{-2} ds \right| = O\left(T^{-1}\right),$$

one concludes that $J = 0$. We consider next the following path defined before;

$$C_{R,T} : 2 - iT \to -R - iT \to -R + iT \to 2 + iT,$$

where R is taken to be a large even integer. By (4.1), one also see that

$$\lim_{T\to\infty} \lim_{R\to\infty} \left(\int_{2-iT}^{2+iT} - \int_{C_{R,T}} \right) \left(-\frac{1}{2\pi i} \cdot \frac{\zeta'(s+1)}{\zeta(s+1)} s^{-2} \right) ds = -\sum_{\rho\in Z} \frac{1}{(\rho-1)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} - b_1.$$

Moreover, the arguments similar to those in the proof of Theorem 1.1 yields

$$\lim_{T\to\infty}\lim_{R\to\infty}\left|\frac{1}{2\pi i}\int_{C_{R,T}}\frac{\zeta'(s+1)}{\zeta(s+1)}s^{-2}ds\right|=0.$$

(In order to show the fact above, we need the several estimates for $\frac{\zeta'(s+1)}{\zeta(s+1)}$. More precisely, see [KW3].) Therefore, since $J=0$ one has

$$\begin{aligned} b_1 &= -\sum_{\rho\in Z}\frac{1}{(\rho-1)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2} \\ &= -\left(\sum_{n=1}^{\infty}\frac{1}{n^2}-1-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\right)-\sum_{\rho\in Z}\frac{1}{\rho^2} \\ &= 1-\frac{3}{4}\zeta(2)-\sum_{\rho\in Z}\frac{1}{\rho^2} \\ &= 1-\frac{\pi^2}{8}-\sum_{\rho\in Z}\frac{1}{\rho^2}. \end{aligned}$$

Proof of (2). We evaluate

$$\lim_{T\to\infty}\left\{-\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{\zeta'(s+1)}{\zeta(s+1)}x^s s^{-2}ds\right\} \tag{4.3}$$

in two ways. For the path $C_{R,T}$ defined in the proof of (1) above, using the residue theorem we obtain

$$\begin{aligned} &\lim_{T\to\infty}\lim_{R\to\infty}\left(\int_{2-iT}^{2+iT}-\int_{C_{R,T}}\right)\left(-\frac{1}{2\pi i}\cdot\frac{\zeta'(s+1)}{\zeta(s+1)}x^s s^{-2}\right)ds \\ =\ & -\sum_{\rho\in Z}\frac{x^{\rho-1}}{(\rho-1)^2}-\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{(2n+1)^2}+\frac{1}{2}(\log x)^2-\gamma\log x-b_1, \end{aligned}$$

since around $s=0$ one has the expansion

$$\begin{aligned} -\frac{\zeta'(s+1)}{\zeta(s+1)}x^s s^{-2} &= -\left(-s^{-1}+\gamma+b_1 s+\cdots\right)\left(1+(\log x)s+\frac{1}{2}(\log x)^2s^2+\cdots\right)s^{-2} \\ &= s^{-3}+(\log x-\gamma)s^{-2}+\left(\frac{1}{2}(\log x)^2-\gamma\log x-b_1\right)s^{-1}+\cdots. \end{aligned}$$

On the other hand, by Lemma 2.3 one sees that (4.3) becomes

$$\begin{aligned} \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s+1}}x^s s^{-2}\right)ds &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{x}{n}\right)^s s^{-2}ds \\ &= \sum_{n<x}\frac{\Lambda(n)}{n}\log\frac{x}{n}. \end{aligned}$$

Therefore we obtain

$$\sum_{n<x}\frac{\Lambda(n)}{n}\log\frac{x}{n}=-\sum_{\rho\in Z}\frac{x^{\rho-1}}{(\rho-1)^2}-\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{(2n+1)^2}+\frac{1}{2}(\log x)^2-\gamma\log x-b_1.$$

It follows that immediately

$$b_1 = \lim_{x\to\infty}\left\{\frac{1}{2}(\log x)^2 - \gamma\log x - \sum_{n<x}\frac{\Lambda(n)}{n}\log\frac{x}{n}\right\}.$$

Proof of (3). The proof is similar to that of (1). In fact, we have

$$-\sum_{\rho\in Z}\frac{1}{(\rho-1)^3}+\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}-b_2=0$$

by considering

$$\lim_{T\to\infty}\left\{-\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{\zeta'(s+1)}{\zeta(s+1)}s^{-3}ds\right\}.$$

Hence

$$\begin{aligned} b_2 &= \sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}-\sum_{\rho\in Z}\frac{1}{(\rho-1)^3} = \left(\sum_{n=1}^{\infty}\frac{1}{n^3}-\sum_{n=1}^{\infty}\frac{1}{(2n)^3}-1\right)+\sum_{\rho\in Z}\frac{1}{\rho^3} \\ &= \frac{7}{8}\zeta(3)-1+\sum_{\rho\in Z}\frac{1}{\rho^3}. \end{aligned}$$

Proof of (4). We study the following limit.

$$\lim_{T\to\infty}\left\{-\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{\zeta'(s+1)}{\zeta(s+1)}x^s s^{-3}ds\right\}. \tag{4.4}$$

By the residue theorem again, we see that the limit (4.4) is equal to

$$-\sum_{\rho\in Z}\frac{x^{\rho-1}}{(\rho-1)^3}+\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{(2n+1)^3}+\frac{1}{6}(\log x)^3-\frac{\gamma}{2}(\log x)^2-b_1\log x-b_2$$

since

$$\begin{aligned} &-\frac{\zeta'(s+1)}{\zeta(s+1)}x^s s^{-3} \\ &= -\left(-s^{-1}+\gamma+b_1 s+b_2 s^2+\cdots\right)\left(1+(\log x)s+\frac{1}{2}(\log x)^2 s^2+\frac{1}{6}(\log x)^3 s^3+\cdots\right)s^{-3} \\ &= \cdots+\left(\frac{1}{6}(\log x)^3-\frac{\gamma}{2}(\log x)^2-b_1\log x-b_2\right)s^{-1}+\cdots \end{aligned}$$

around $s=0$. On the other hand, using lemma 2.3 again for $n=2$ one finds that (4.4) is equal to

$$\sum_{n<x}\frac{\Lambda(n)}{n}\cdot\frac{1}{2}\left(\log\frac{x}{n}\right)^2.$$

Therefore, we have

$$\begin{aligned} &\sum_{n<x}\frac{\Lambda(n)}{n}\cdot\frac{1}{2}\left(\log\frac{x}{n}\right)^2 = \\ &\qquad -\sum_{\rho\in Z}\frac{x^{\rho-1}}{(\rho-1)^3}+\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{(2n+1)^3}+\frac{1}{6}(\log x)^3-\frac{\gamma}{2}(\log x)^2-b_1\log x-b_2, \end{aligned}$$

whence it follows that

$$b_2=\lim_{x\to\infty}\left\{\frac{1}{6}(\log x)^3-\frac{1}{2}\gamma(\log x)^2-b_1\log x-\frac{1}{2}\sum_{n<x}\frac{\Lambda(n)}{n}\left(\log\frac{x}{n}\right)^2\right\}.$$

These complete the whole of the proof of Theorem B. ∎

Remark 4.1. Clearly, equating the two expressions (1) and (2) of Theorem 3, one arrives at the following identity. that

$$\sum_{\rho \in Z} \frac{1}{\rho^2} = 1 - \frac{\pi^2}{8} - \lim_{x \to \infty} \left\{ \frac{1}{2} (\log x)^2 - \gamma \log x - \sum_{n<x} \frac{\Lambda(n)}{n} \log \frac{x}{n} \right\}.$$

Furthermore, by equating (3) and (4) and using (2) again one has

$$\sum_{\rho \in Z} \frac{1}{\rho^3} = 1 - \frac{7}{8}\zeta(3) + \lim_{x \to \infty} \left\{ -\frac{1}{3} (\log x)^3 + \frac{1}{2}\gamma (\log x)^2 + \frac{1}{2} \sum_{n<x} \frac{\Lambda(n)}{n} (\log \frac{x}{n})(\log nx) \right\}.$$

Note here that $\zeta(3)$ is expressed as

$$\frac{7}{8}\zeta(3) = \pi^2 \left\{ \frac{1}{4} \log 2 - \frac{1}{8} + \sum_{m=1}^{\infty} \frac{\zeta(2m)}{(m+1)^{4m+1}} \right\}. \quad \text{(see [KW1])}$$

Similar analysis we developed here enables us to obtain in principle the formulas for further higher powers of essential zeroes.

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Y. Iijima,
Department of Mathematics, Tokyo Institute of Technology,
Meguro, Tokyo, 152-0033 JAPAN
Email address: `iijima@math.titech.ac.jp`

Y. Hashimoto,
Graduate School of Mathematics, Kyushu University,
Hakozaki, Fukuoka, 812-8581 JAPAN
Email address: `hasimoto@math.kyushu-u.ac.jp`

N. Kurokawa,
Zeta Institute & Department of Mathematics,
Tokyo Institute of Technology,
Meguro, Tokyo, 152-0033 JAPAN
Email address: `kurokawa@math.titech.ac.jp`

M. Wakayama,
Zeta Institute & Faculty of Mathematics,
Kyushu University, Hakozaki, Fukuoka, 812-8581 JAPAN
Email address: `wakayama@math.kyushu-u.ac.jp`