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Injective spectra with respect to the K-homologies

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§0. Introduction

Given a CW-spectrum E we call a CW-spectrum W *E_*-injective* if any map $f: X \to Y$ induces an epimorphism $f^*: [Y, W] \to [X, W]$ whenever $f_*: E_*X \to E_*Y$ is a monomorphism [12, Definition 1 i)]. The well known ring spectra $E = S, HZ/p, MO, MU, MS_p, KU, KO$ and KT satisfy some of nice properties as stated in [1] or [2]. For example, E_*E is flat as an E_*-module, and the product map $v_{E,F}: E_*E \otimes_{E_*} \pi_*F \to E_*F$ is an isomorphism for any E-module spectrum F. Then E_*X may be regarded as a comodule over the coalgebra E_*E. For such a nice ring spectrum E we gave the following characterization in [17].

Theorem 1. *Let E be a ring spectrum satisfying the above two properties. For a CW-spectrum W the following conditions are equivalent*:
i) *W is an E_*-injective spectrum,*
ii) *W is an E_*-local spectrum such that E_*W is injective as an E_*E-comodule, and*
iii) *the canonical morphism $\kappa_E: [X, W] \longrightarrow \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X.*

In this note we study $K_*^{\mathscr{C}}$-injective spectra for $K^{\mathscr{C}} = KU \vee KO \vee KT$, $KU \vee KO$, $KU \vee KT$, $KO \vee KT$, KU, KO and KT where KU, KO and KT denote the complex, the real and the self-conjugate K-spectrum respectively. In particular, we give a $K_*^{\mathscr{C}}$-version of Theorem 1 as our main result (see Theorem 2 below). For our purpose we use the Bousfield's abelian categories CRT and $ACRT$ [9, 2.1 and 5.5] whose objects $M = \{M^C, M^R, M^T\}$ are modelled on the united K-homologies $K_*^{CRT}X = \{KU_*X, KO_*X, KT_*X\}$ for any CW-spectra X, although our category $ACRT$ is somewhat different from the Bousfield's one.

In §1 we first recall the abelian category CRT and then state several homological properties of CRT established in [9, §2 and §3] for later use. In §2 we introduce the abelian categories $\mathscr{C} = CR, CT, RT, C, R$ and T whose objects $M = \{M^H\}$ are obtained by restricting their namesakes in CRT, of which CR and C have already been done in [9, 4.1 and 4.7]. In §3 we give simple criteria for projective and injective objects in $\mathscr{C}$ (Theorems 3.1, 3.2, 3.3 and 3.4) by referring the CRT cases demonstrated in [9, Theorems 3.2 and 3.3].

In §4 we show that $KU \wedge SZ/2^\infty$, $KO \wedge SZ/2^\infty$ and $KT \wedge SZ/2^\infty$ respectively are never $(KO \vee KT)_*$-, $(KU \vee KT)_*$- and $(KU \vee KO)_*$-injective where $SZ/2^\infty$ denotes the Moore spectrum of type $Z/2^\infty$ (Lemma 4.3). This result gives necessary and sufficient conditions under which a CW-spectrum W is $K_*^{\mathscr{C}}$-injective (Theorems 4.7 and 4.9). In §5 we introduce the abelian categories $A\mathscr{C}$ consisting of objects M of $\mathscr{C}$ having a KO_*KO-comodule structure when $\mathscr{C} = CRT, CR, CT, RT, C, R$ and T, as the united K-homology $K_*^{CRT}X$ admits a KO_*KO-comodule structure. Although our category $A\mathscr{C}$ is not the quite same as the abelian category $A\mathscr{C}$ consisting of objects M of $\mathscr{C}$ with stable Adams operations introduced in [9, 5.5], we use the same notation as Bousfield's (see [8, §10]). In fact we can show the same result (Theorem 5.2) that an object M in our category $ACRT$ has injective dimension ≤ 2 as [9, Theorem 7.3] in Bousfield's category $ACRT$. We finally give the following characterization as our main result (Theorem 5.8).

THEOREM 2. *Let $\mathscr{C}$ denotes one of the abelian categories CRT, CR, CT, RT, C, R and T. For a CW-spectrum W the following conditions are equivalent*:
i) *W is a $K_*^{\mathscr{C}}$-injective spectrum,*
ii) *W is a quasi KO-module spectrum such that $K_*^{\mathscr{C}}W$ is injective in $\mathscr{C}$,*
iii) *W is a KO_*-local spectrum such that $K_*^{\mathscr{C}}W$ is injective in $A\mathscr{C}$, and*
iv) *the canonical morphism $\kappa_K^{\mathscr{C}}: [X, W] \to \mathrm{Hom}_{A\mathscr{C}}(K_*^{\mathscr{C}}X, K_*^{\mathscr{C}}W)$ is an isomorphism for any CW-spectrum X.*

Here a *quasi* KO-module spectrum W is meant a KO-module spectrum which is not necessarily assumed to be associative.

§1. The Bousfield's abelian category CRT

1.1. Let KU, KO and KT denote the complex, the real and the self-conjugate K-spectrum respectively. In [9] KU is denoted by K and in [5, 15 and 16] KT is denoted by KC. All of these periodic K-spectra are commutative ring spectra and their coefficient rings are represented as follows:

$$\begin{aligned}
&\pi_*KU \cong Z[B_C, B_C^{-1}]\\
(1.1)\qquad &\pi_*KO \cong Z[B_R, B_R^{-1}, \eta_R, \xi]/\{2\eta_R = 0, \eta_R^3 = 0, \xi^2 = 4B_R, \eta_R\xi = 0\}\\
&\pi_*KT \cong Z[B_T, B_T^{-1}, \eta_T, \omega]/\{2\eta_T = 0, \eta_T^2 = 0, \omega^2 = 0, \eta_T\omega = 0\}
\end{aligned}$$

where $B_C \in \pi_2KU \cong Z$, $B_R \in \pi_8KO \cong Z$, $B_T \in \pi_4KT \cong Z$, $\eta_R \in \pi_1KO \cong Z/2$, $\eta_T \in \pi_1KT \cong Z/2$, $\xi \in \pi_4KO \cong Z$ and $\omega \in \pi_3KT \cong Z$. These coefficient rings

$\pi_* KU$, $\pi_* KO$ *and* $\pi_* KT$ will often be abbrebiated as KU_*, KO_* and KT_* respectively.

We denote by $P = C(\eta)$ and $Q = C(\eta^2)$ the cofibers of the stable Hopf map $\eta\colon \Sigma^1 \to \Sigma^0$ of order 2 and its square $\eta^2\colon \Sigma^2 \to \Sigma^0$ respectively. The complex K-spectrum KU and the self-conjugate K-spectrum KT have standard decompositions $KU \cong KO \wedge P$ and $KT \cong KO \wedge Q$ as KO-module spectra ([5] or [15]). Hereafter we shall often identify the periodic K-spectra KU and KT with the smash products $KO \wedge P$ and $KO \wedge Q$ respectively. Since the elementary spectra P and Q are self-dual [13], there exist duality isomorphisms

$$\text{(1.2)} \quad \begin{aligned} D_P&\colon [\Sigma^2 X, KU \wedge Y] \to [P \wedge X, KO \wedge Y] \\ D_Q&\colon [\Sigma^3 X, KT \wedge Y] \to [Q \wedge X, KO \wedge Y] \end{aligned}$$

for any CW-spectra X and Y.

As relations among the periodic K-spectra KU, KO and KT we have Anderson's cofiber sequences ([5] or [9]):

$$\text{i)} \quad \Sigma^1 KO \xrightarrow{\eta \wedge 1} KO \xrightarrow{c} KU \xrightarrow{rB_C^{-1}} \Sigma^2 KO$$

$$\text{ii)} \quad \Sigma^2 KO \xrightarrow{\eta^2 \wedge 1} KO \xrightarrow{\varepsilon} KT \xrightarrow{\tau B_T^{-1}} \Sigma^3 KO$$

$$\text{(1.3)} \quad \text{iii)} \quad KT \xrightarrow{\zeta} KU \xrightarrow{B_C^{-1}(1-\psi_C^{-1})} \Sigma^2 KU \xrightarrow{\gamma B_C} \Sigma^1 KT$$

$$\text{iv)} \quad \Sigma^1 KT \xrightarrow{(-\tau, \tau B_T^{-1})} KO \vee \Sigma^4 KO \xrightarrow{c \vee B_C^2 c} KU \xrightarrow{\varepsilon r B_C^{-1}} \Sigma^2 KT$$

$$\text{v)} \quad \Sigma^2 KU \xrightarrow{(-rB_C, rB_C^{-1})} KO \vee \Sigma^4 KO \xrightarrow{\varepsilon \vee B_T \varepsilon} KT \xrightarrow{c\tau B_T^{-1}} \Sigma^3 KU.$$

Here $\eta\colon \Sigma^1 \to \Sigma^0$ denotes the stable Hopf map of order 2, $B_C\colon \Sigma^2 KU \to KU$, $B_R\colon \Sigma^8 KO \to KO$ and $B_T\colon \Sigma^4 KT \to KT$ the periodicity maps, and $\psi_C^{-1}\colon KU \to KU$ and $\psi_T^{-1}\colon KT \to KT$ the conjugation maps which are ring maps with $\psi_C^{-1}\psi_C^{-1} = 1$ and $\psi_T^{-1}\psi_T^{-1} = 1$. The maps $c\colon KO \to KU$, $\varepsilon\colon KO \to KT$ and $\zeta\colon KT \to KU$ are ring maps with $c = \zeta\varepsilon$, and the maps $r\colon KU \to KO$, $\tau\colon \Sigma^1 KT \to KO$ are merely KO-module maps and $\gamma\colon KU \to \Sigma^1 KT$ is a KT-module map with $r = \tau\gamma$.

Let E be a ring spectrum and F be an E-module spectrum equipped with a structure map $\mu\colon F \wedge E \to F$. For any CW-spectra X and Y we consider the homomorphism

$$\kappa_n^F\colon [X, E \wedge Y] \to \mathrm{Hom}(F_n X, F_n Y)$$

assigning to each map $f\colon X \to E \wedge Y$ the induced homomorphism $\kappa^F(f)_*$ in dimension n where $\kappa^F(f) = (\mu \wedge 1)(1 \wedge f)\colon F \wedge X \to F \wedge Y$. For an abelian group G we denote by SG the Moore spectrum of type G. Then there exist universal coefficient sequences for the periodic K-spectra KU, KO and KT ([6] or [14]):

$$\text{i)}\quad 0 \to \mathrm{Ext}(KU_5X, G) \to [X, KU \wedge SG] \xrightarrow{\kappa_6^{KU}} \mathrm{Hom}(KU_6X, G) \to 0$$

$$(1.4)\quad \text{ii)}\quad 0 \to \mathrm{Ext}(KO_3X, G) \to [X, KO \wedge SG] \xrightarrow{\kappa_4^{KO}} \mathrm{Hom}(KO_4X, G) \to 0$$

$$\text{iii)}\quad 0 \to \mathrm{Ext}(KT_6X, G) \to [X, KT \wedge SG] \xrightarrow{\kappa_7^{KT}} \mathrm{Hom}(KT_7X, G) \to 0$$

in which all of KU_6SG, KO_4SG and KT_7SG are identified with G.

For any KO-module spectra W and Z we denote by $[W, Z]_{KO}$ the subgroup of $[W, Z]$ consisting of all the homotopy classes of KO-module maps. Consider the homomorphisms

$$\text{i)}\quad \tilde{\kappa}_6^{KU}\colon [W, KU \wedge SG]_{KO} \to \mathrm{Hom}(\pi_6 P \wedge W, G)$$

$$(1.5)\quad \text{ii)}\quad \tilde{\kappa}_4^{KO}\colon [W, KO \wedge SG]_{KO} \to \mathrm{Hom}(\pi_4 W, G)$$

$$\text{iii)}\quad \tilde{\kappa}_7^{KT}\colon [W, KT \wedge SG]_{KO} \to \mathrm{Hom}(\pi_7 Q \wedge W, G)$$

defined by $\tilde{\kappa}_6^{KU}(f) = (\mu_P \wedge 1)_*(1 \wedge f)_*$, $\tilde{\kappa}_4^{KU}(g) = g_*$ and $\tilde{\kappa}_7^{KT}(h) = (\mu_Q \wedge 1)_*(1 \wedge h)_*$ where $\mu_P\colon P \wedge KU \to KU$ and $\mu_Q\colon Q \wedge KT \to KT$ are associated with the multiplications of KU and KT. As is easily checked, the above $\tilde{\kappa}_6^{KU}$, $\tilde{\kappa}_4^{KO}$ and $\tilde{\kappa}_7^{KT}$ are isomorphisms for any KO-module spectrum W whenever the abelian group G is divisible.

1.2. We first recall the abelian category CRT introduced by Bousfield [9, 2.1]. For any CW-spectrum X the united K-homology

$$K_*^{CRT}X = \{KU_*X, KO_*X, KT_*X\}$$

is just viewed as a model of an object of the abelian category CRT. An object CRT is a triple $M = \{M^C, M^R, M^T\}$ consisting of a KU_*-, KO_*- and KT_*-module M^C, M^R and M^T equipped with operations below (1.6). Thus M^C, M^R and M^T are united by the following KO_*-module maps called operations:

$$(1.6)\quad \begin{aligned}
&B_C\colon \Sigma^2 M^C \xrightarrow{\simeq} M^C,\ \psi_C^{-1}\colon M^C \xrightarrow{\simeq} M^C,\ B_R\colon \Sigma^8 M^R \xrightarrow{\simeq} M^R,\\
&\eta_R\colon \Sigma^1 M^R \longrightarrow M^R,\ \xi\colon \Sigma^4 M^R \longrightarrow M^R,\ B_T\colon \Sigma^4 M^T \xrightarrow{\simeq} M^T,\\
&\eta_T\colon \Sigma^1 M^T \longrightarrow M^T,\ \omega\colon \Sigma^3 M^T \longrightarrow M^T,\ \psi_T^{-1}\colon M^T \xrightarrow{\simeq} M^T,\\
&\varepsilon\colon M^R \longrightarrow M^T,\ \tau\colon \Sigma^1 M^T \longrightarrow M^R,\ \zeta\colon M^T \longrightarrow M^C \text{ and } \gamma\colon M^C \longrightarrow \Sigma^1 M^T
\end{aligned}$$

where $(\Sigma^n M)_* = M_{*-n}$, which satisfy the following relations listed in [9, 1. 9]:

(1.7) i) $2\eta_R = 0,\ \eta_R^3 = 0,\ \xi^2 = 4B_R,\ \eta_R\xi = 0,$

ii) $2\eta_T = 0,\ \eta_T^2 = 0,\ \omega^2 = 0,\ \eta_T\omega = 0,$

iii) $\psi_C^{-1}\psi_C^{-1} = 1,\ \psi_C^{-1}B_C = -B_C\psi_C^{-1},$

iv) $\psi_T^{-1}\psi_T^{-1} = 1,\ \psi_T^{-1}B_T = B_T\psi_T^{-1},$

v) $f\eta_H = \eta_L f$ for any map $f\colon M^H \to M^L$ where $H, L \in \{C, R, T\}$ and $\eta_C = 0,$

vi) $\varepsilon B_R = B_T^2\varepsilon,\ \tau B_T^2 = B_R\tau,\ \psi_T^{-1}\varepsilon = \varepsilon,\ \tau\psi_T^{-1} = -\tau,$

vii) $\zeta B_T = B_C^2\zeta,\ \gamma B_C^2 = B_T\gamma,\ \psi_C^{-1}\zeta = \zeta\psi_T^{-1} = \zeta,\ \psi_T^{-1}\gamma = -\gamma\psi_C^{-1} = -\gamma,$

viii) $\tau\varepsilon = \eta_R,\ \tau B_T\varepsilon = 0,\ \xi = \tau\omega\varepsilon,\ B_T\varepsilon\tau B_T^{-1} = \varepsilon\tau + \eta_T,$

ix) $\zeta\gamma = 0,\ \gamma B_C\zeta = \eta_T,\ \omega = B_T\gamma\zeta$ and

x) $\zeta\varepsilon\tau\gamma = 1 + \psi_C^{-1},\ \tau\gamma\zeta\varepsilon = 2,\ \varepsilon\tau\gamma\zeta = 1 + \psi_T^{-1},\ \gamma\zeta\varepsilon\tau = 1 - \psi_T^{-1}.$

A morphism of CRT is a triple $f = \{f^C, f^R, f^T\}$ consisting of a KU_*-, KO_*- and KT_*-module map f^C, f^R and f^T which commute the above operations.

An object $M = \{M^C, M^R, M^T\}$ of CRT is called *CRT-acyclic* [9, 2.3] when the three sequences

(1.8) i) $\cdots \longrightarrow \Sigma^1 M^R \xrightarrow{\eta_R} M^R \xrightarrow{c} M^C \xrightarrow{rB_C^{-1}} \Sigma^2 M^R \xrightarrow{\eta_R} \Sigma^1 M^R \longrightarrow \cdots$

ii) $\cdots \longrightarrow \Sigma^2 M^R \xrightarrow{\eta_R^2} M^R \xrightarrow{\varepsilon} M^T \xrightarrow{\tau B_T^{-1}} \Sigma^3 M^R \xrightarrow{\eta_R^2} \Sigma^1 M^R \longrightarrow \cdots$

iii) $\cdots \longrightarrow M^T \xrightarrow{\zeta} M^C \xrightarrow{B_C^{-1}(1-\psi_C^{-1})} \Sigma^2 M^C \xrightarrow{\gamma B_C} \Sigma^1 M^T \xrightarrow{\zeta} \Sigma^1 M^C \longrightarrow \cdots$

become exact in which $c = \zeta\varepsilon$ and $r = \tau\gamma$. For each KO-module spectrum W the united homotopy $\pi_*^{CRT} W = \{\pi_* P \wedge W, \pi_* W, \pi_* Q \wedge W\}$ is viewd as an object of CRT and it is always CRT-acyclic. Obviously $\pi_*^{CRT} KO \wedge X \cong K_*^{CRT} X$ for any CW-spectrum X.

The united K-homologies of the elementary spectra Σ^0, $P = C(\eta)$ and $Q = C(\eta^2)$ are represented as follows [9, 2.4]:

(1.9) i) $KO_0\Sigma^0 \cong Z\{b\},\ KO_1\Sigma^0 \cong Z/2\{\eta_R b\},\ KO_2\Sigma^0 \cong Z/2\{\eta_R^2 b\},$

$KO_4\Sigma^0 \cong Z\{\xi b\},\ KO_n\Sigma^0 = 0$ for $n =$ 3, 5, 6 and 7,

$KU_0\Sigma^0 \cong Z\{cb\},\ KU_1\Sigma^0 = 0,\ KT_0\Sigma^0 \cong Z\{\varepsilon b\},$

$KT_1\Sigma^0 \cong Z/2\{\eta_T\varepsilon b\}$, $KT_2\Sigma^0 = 0$ and $KT_3\Sigma^0 \cong Z\{\omega\varepsilon b\}$.

ii) $KU_2P \cong Z\{b\} \oplus Z\{\psi_C^{-1}b\}$, $KU_3P = 0$, $KO_2P \cong Z\{rb\}$, $KO_3P = 0$,

$KT_2P \cong Z\{\varepsilon\tau\gamma b\}$ and $KT_3P \cong Z\{\gamma B_C b\}$.

iii) $KT_3Q \cong Z\{b\} \oplus Z\{\psi_T^{-1}b\}$, $KT_4Q \cong Z\{\varepsilon\tau b\} \oplus Z/2\{\eta_T b\}$,

$KT_5Q \cong Z/2\{\eta_T\varepsilon\tau b\}$, $KT_6Q \cong Z\{\omega b\}$, $KU_3Q \cong Z\{\zeta b\}$,

$KU_4Q \cong Z\{\zeta\varepsilon\tau b\}$, $KO_3Q \cong Z\{\tau\gamma\zeta b\}$, $KO_4Q \cong Z\{\tau b\}$,

$KO_5Q \cong Z/2\{\eta_R\tau b\}$ and $KO_6Q = 0$.

For a graded abelian group $G = \{G_n\}$ we denote by SG the wedge sum $\vee_n \Sigma^n SG_n$ of the suspended Moore spectra. A KO-module spectrum W is said to be π_*^{CRT}*-free* if it is expressed as a wedge sum of copies of the KO-module spectra $\Sigma^n KU$, $\Sigma^n KO$ and $\Sigma^n KT$. A KO-module spectrum W is said to be a π_*^{CRT}*-cofree* if it is expressed as a wedge sum of KO-module spectra $KU \wedge SA$, $KO \wedge SB$ and $KT \wedge SC$ where $A = \{A_i\}_{0\le i\le 1}$, $B = \{B_j\}_{0\le j\le 7}$ and $C = \{C_k\}_{0\le k\le 3}$ are graded divisible. A *free* object of CRT is isomorphic in CRT to a certain united homotopy $\pi_*^{CRT}W$ with W π_*^{CRT}-free, and dually a *cofree object of* CRT is isomorphic in CRT to $\pi_*^{CRT}W$ with W π_*^{CRT}-cofree.

1.3. We now recall several homological properties of the abelian category CRT investigated in [9, §2 and §3]. Given an object $N = \{N^C, N^R, N^T\}$ of CRT we define three homomorphisms

$$\text{(1.10)}\qquad \begin{aligned} &\text{i)}\ e_C\colon \mathrm{Hom}_{CRT}(K_{*+2}^{CRT}P, N) \longrightarrow N_0^C \\ &\text{ii)}\ e_R\colon \mathrm{Hom}_{CRT}(K_*^{CRT}\Sigma^0, N) \longrightarrow N_0^R \\ &\text{iii)}\ e_T\colon \mathrm{Hom}_{CRT}(K_{*+3}^{CRT}Q, N) \longrightarrow N_0^T \end{aligned}$$

by $e_C(f) = f^C(b_C)$, $e_R(f) = f^R(b_R)$ and $e_T(f) = f^T(b_T)$ for each morphism $f = \{f^C, f^R, f^T\}$ of CRT where $b_C \in KU_2P$, $b_R \in KO_0\Sigma^0$ and $b_T \in KT_3Q$ denote the standard generators b given in (1.9). Because of (1.9) the united K-homologies $K_*^{CRT}P$, $K_*^{CRT}\Sigma^0$ and $K_*^{CRT}Q$ are determined completely by the standard generators b_C, b_R and b_T. Hence we can easily see

LEMMA 1.1. *The above e_C, e_R and e_T are all isomorphisms for any object N of CRT.*

By means of Lemma 1.1 there exists a π_*^{CRT}-free spectrum $KO \wedge X$ and an epimorphism $f\colon K_*^{CRT}X \to N$ in CRT for each object N of CRT. Thus the abelian category CRT has enough projectives. In [9, Theorem 3.2] Bousfield has established a simple criterion for projective objects in CRT.

THEOREM 1.2 [9, Theorem 3.2]. *For an object $M=\{M^C, M^R, M^T\}$ of the abelian category CRT the following conditions are equivalent*:
i) *M is projective in CRT,*
ii) *M is CRT-acyclic with M^C free, and*
iii) *M is free in CRT, thus it is isomorphic in CRT to a direct sum $K_*^{CRT}P \wedge SA \oplus K_*^{CRT}SB \oplus K_*^{CRT}Q \wedge SC$ where $A=\{A_i\}_{0\le i\le 1}$, $B=\{B_j\}_{0\le j\le 7}$ and $C=\{C_k\}_{0\le k\le 3}$ are graded free.*

Given an object $M=\{M^C, M^R, M^T\}$ of CRT and an abelian group G we define three homomorphisms

$$\text{(1.11)}\qquad \begin{aligned} &\text{i)}\quad \varphi_C\colon \mathrm{Hom}_{CRT}(M, K_{*+6}^{CRT}P \wedge SG) \longrightarrow \mathrm{Hom}(M_0^C, G)\\ &\text{ii)}\quad \varphi_R\colon \mathrm{Hom}_{CRT}(M, K_{*+4}^{CRT}SG) \longrightarrow \mathrm{Hom}(M_0^R, G)\\ &\text{iii)}\quad \varphi_T\colon \mathrm{Hom}_{CRT}(M, K_{*+7}^{CRT}Q \wedge SG) \longrightarrow \mathrm{Hom}(M_0^T, G)\end{aligned}$$

by $\varphi_C(f)=(\mu_P\wedge 1)_* f_0^C$, $\varphi_R(f)=f_0^R$ and $\varphi_T(f)=(\mu_Q\wedge 1)_* f_0^T$ for each morphism $f=\{f^C, f^R, f^T\}$ of CRT. Here $\mu_P\colon KU\wedge P\to KU$ and $\mu_Q\colon KT\wedge Q\to KT$ are associated with the multiplications of KU and KT, and all of KU_6SG, KO_4SG and KT_7SG are identified with G as in (1.4).

When $M=K_{*+2}^{CRT}\Sigma^nP$, $K_*^{CRT}\Sigma^n$ or $K_{*+3}^{CRT}\Sigma^nQ$, the above φ_C, φ_R and φ_T admit the following factorizations:

$$\begin{aligned}
&\varphi_C=\kappa_6^{KU}D_Pe_C,\ \varphi_R=\kappa_4^{KO}D_Pe_R \ \text{ and }\ \varphi_T=\kappa_7^{KT}D_Pe_T \ \text{ when }\ M=K_{*+2}^{CRT}\Sigma^nP,\\
&\varphi_C=\kappa_6^{KU}e_C\ \ ,\ \varphi_R=\kappa_4^{KO}e_R \ \text{ and }\ \varphi_T=\kappa_7^{KT}e_T \ \text{ when }\ M=K_*^{CRT}\Sigma^n,\\
&\varphi_C=\kappa_6^{KU}D_Qe_C,\ \varphi_R=\kappa_4^{KO}D_Qe_R \ \text{ and }\ \varphi_T=\kappa_7^{KT}D_Qe_T \ \text{ when }\ M=K_{*+3}^{CRT}\Sigma^nQ.
\end{aligned}$$

Here e_C, e_R and e_T are the isomorphisms defined in (1.10), κ_6^{KU}, κ_4^{KO} and κ_7^{KT} are the epimorphisms appeared in (1.4) and D_p and D_Q are the duality isomorphisms given in (1.2). By virtue of these factorizations we can easily show

LEMMA 1.3. *For any object M of CRT, the above φ_C is always an isomorphism, and both of φ_R and φ_T are isomorphisms if the abelian group G is 2-divisible* (see [9, 2.5]).

This implies that the united K-homologies $K_*^{CRT}\Sigma^nP\wedge SG$, $K_*^{CRT}\Sigma^nSG$ and $K_*^{CRT}\Sigma^nQ\wedge SG$ are injective in CRT whenever G is divisible. Moreover there exists a π_*^{CRT}-cofree spectrum $KO\wedge Y$ and a monomorphism $f\colon M\to K_*^{CRT}Y$ in CRT for each object M of CRT. Thus the abelian category CRT has enough injectives, too. In [9, Theorem 3.3] Bousfield has established a simple criterion for injective objects in CRT as a dual of Theorem 1.2.

THEOREM 1.4 [9, Theorem 3.3]. *For an object* $M=\{M^C, M^R, M^T\}$ *of the abelian category* CRT *the following conditions are equivalent*:
i) M *is injective in* CRT,
ii) M *is* CRT*-acyclic with* M^C *divisible, and*
iii) M *is cofree in* CRT, *thus it is isomorphic in* CRT *to a direct sum* $K_*^{CRT}P \wedge SA \oplus K_*^{CRT}SB \oplus K_*^{CRT}Q \wedge SC$ *where* $A=\{A_i\}_{0\le i\le 1}$, *and* $C=\{C_k\}_{0\le k\le 3}$ *are graded divisible 2-torsion and* $B=\{B_j\}_{0\le j\le 7}$ *is graded divisible.*

1.4. For any CW-spectra X and Y we consider the homomorphism

$$\kappa^{CRT}\colon [X, KO \wedge Y] \longrightarrow \mathrm{Hom}_{CRT}(K_*^{CRT}X, K_*^{CRT}Y)$$

assingning to each map $f\colon X\to KO\wedge Y$ the induced homomorphism $\kappa^{CRT}(f) = \{\kappa^{KU}(f)_*, \kappa^{KO}(f)_*, \kappa^{KT}(f)_*\}$ where $\kappa^K(f)=(\mu\wedge 1)(1\wedge f)\colon K\wedge X\to K\wedge Y$ for $K=KU$, KO or KT. Compose φ_C, φ_R or φ_T given in (1.11) after the above κ^{CRT} when $Y=P\wedge SG$, SG or $Q\wedge SG$. Then it is easily checked that $\kappa_6^{KU}=\varphi_C\kappa^{CRT}$, $\kappa_4^{KO}=\varphi_R\kappa^{CRT}$ and $\kappa_7^{KT}=\varphi_T\kappa^{CRT}$ where κ_6^{KU}, κ_4^{KO} and κ_7^{KT} are appeared in (1.4). For any KO-module spectrum W we next compose φ_C, φ_R or φ_T after the canonical homomorphism

$$\kappa_S^{CRT}\colon [W, KO\wedge Y]_{KO} \longrightarrow \mathrm{Hom}_{CRT}(\pi_*^{CRT}W, K_*^{CRT}Y)$$

when $Y=P\wedge SG$, SG or $Q\wedge SG$. Then it is immediate that $\tilde\kappa_6^{KU}=\varphi_C\kappa_S^{CRT}$, $\tilde\kappa_4^{KO}=\varphi_R\kappa_S^{CRT}$ and $\tilde\kappa_7^{KT}=\varphi_T\kappa_S^{CRT}$ where $\tilde\kappa_6^{KU}$, $\tilde\kappa_4^{KU}$ and $\tilde\kappa_7^{KT}$ are appeared in (1.5).

Hence Lemma 1.3 combined with (1.4) and (1.5) implies immediately

LEMMA 1.5. *If a* KO*-module spectrum* Z *is* π_*^{CRT}*-cofree, then*
i) $\kappa^{CRT}\colon [X, Z]\to \mathrm{Hom}_{CRT}(K_*^{CRT}X, \pi_*^{CRT}Z)$ *is an isomorphism for any* CW*-spectrum* X, *and*
ii) $\kappa_S^{CRT}\colon [W, Z]_{KO}\to \mathrm{Hom}_{CRT}(\pi_*^{CRT}W, \pi_*^{CRT}Z)$ *is an isomorphism for any* KO*-module spectrum* W.

Using Theorem 1.4 and Lemma 1.5 ii) we show

LEMMA 1.6. *For each* KO*-module spectrum* W *there exist* π_*^{CRT}*-cofree spectra* Z_0 *and* Z_1 *and a cofiber sequence* $W\to Z_0\to Z_1$ *of* KO*-module spectra inducing a short exact sequence* $0\to\pi_*^{CRT}W\to\pi_*^{CRT}Z_0\to\pi_*^{CRT}Z_1\to 0$ *in* CRT.

PROOF. Choose divisible abelian groups $A_i(0\le i\le 1)$, $B_j(0\le j\le 7)$ and $C_k(0\le k\le 3)$ so that $\pi_iP\wedge W$, $\pi_{j+4}W$ and $\pi_{k+3}Q\wedge W$ are embedded into A_i, B_j and C_k respectively. Setting $Z_0=KU\wedge SA\vee KO\wedge SB\vee KT\wedge SC$ with $A=\{A_i\}$, $B=\{B_j\}$ and $C=\{C_k\}$, we get a KO-module map $f\colon W\to Z_0$ which induces a given monomorphism $f_*\colon \pi_*^{CRT} W \to \pi_*^{CRT} Z_0$ by means of Lemma 1.5 ii). Consider the cofiber sequence $W \xrightarrow{f} Z_0 \xrightarrow{h} Y$. Then the united homotopy $\pi_*^{CRT} Y$ is an acyclic object of CRT with $\pi_* P \wedge Y$ divisible, although the cofiber Y might not possess a KO-module structure which is associative. According to Theorem 1.4 $\pi_*^{CRT} Y$ is isomorphic in CRT to a certain united homotopy $\pi_*^{CRT} Z_1$ with Z_1 π_*^{CRT}-cofree. Using Lemma 1.5 ii) again we get a KO-module map $g : Z_0 \to Z_1$ whose induced homomorphism $g_* : \pi_*^{CRT} Z_0 \to \pi_*^{CRT} Z_1$ coincides with the epimorphism $h_*\colon \pi_*^{CRT} Z_0 \to \pi_*^{CRT} Y$ when $\pi_*^{CRT} Z_1$ is identified with $\pi_*^{CRT} Y$. Since the composite map $gf\colon W \to Z_0 \to Z_1$ is trivial, there exists a map $k\colon Y \to Z_1$ with $kh = g$, which is in fact an equivalence. Thus we have a cofiber sequence $W \to Z_0 \to Z_1$ of KO-module spectra as desired.

Puttting Lemma 1.5 i) and 1.6 together we can easily construct the universal coefficient sequence given in [9, 9.6].

THEOREM 1.7. *For a CW-spectrum X and a KO-module spectrum W, there exists a natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}_{CRT}(K_*^{CRT} X, \pi_{*+1}^{CRT} W) \longrightarrow [X, W] \longrightarrow \mathrm{Hom}_{CRT}(K_*^{CRT} X, \pi_*^{CRT} W) \longrightarrow 0.$$

Combining Theorems 1.2 and 1.4 with Theorem 1.7 we immediately obtain

THEOREM 1.8 *Let W be a KO-module spectrum.*

i) *If $\pi_* P \wedge W$ is free, then W is π_*^{CRT}-free, thus it is isomorphic as KO-module spectra to a certain wedge sum $KU \wedge SA \vee KO \wedge SB \vee KT \wedge SC$ where $A = \{A_i\}_{0 \le i \le 1}$, $B = \{B_j\}_{0 \le j \le 7}$ and $C = \{C_k\}_{0 \le k \le 3}$ are graded free.*

ii) *If $\pi_* P \wedge W$ is divisible, then W is π_*^{CRT}-cofree, thus it is isomorphic as KO-module spectra to a certain wedge sum $KU \wedge SA \vee KO \wedge SB \vee KT \wedge SC$ where $A = \{A_i\}_{0 \le i \le 1}$ and $C = \{C_k\}_{0 \le k \le 3}$ are graded divisible 2-torsion and $B = \{B_j\}_{0 \le j \le 7}$ is graded divisible.*

See [16, Theorems 2.4 and 3.4] for a direct proof.

§2. The abelian categories $\mathscr{C}$ derived from CRT

2.1. By replacing the united K-homologies $K_*^{CRT} X = \{KU_* X, KO_* X, KT_* X\}$ by the simpler K-homology object $K_*^{CR} X = \{KU_* X, KO_* X\}$, $K_*^{CT} X = \{KU_* X, KT_* X\}$ or $K_*^{RT} X = \{KO_* X, KT_* X\}$, we here introduce new abelian categories CR, CT and RT.

An object of CR is a pair $M = \{M^C, M^R\}$ consisting of a KU_*- and KO_*-module M^C and M^R equipped with the operations below (see [9, 3.7]). Thus M^C and M^R are united by the following KO_*-module maps:

$$B_C\colon \Sigma^2 M^C \xrightarrow{\simeq} M^C,\ \psi_C^{-1}\colon M^C \xrightarrow{\simeq} M^C,\ B_R\colon \Sigma^8 M^R \xrightarrow{\simeq} M^R,\ \eta_R\colon \Sigma^1 M^R \to M^R$$

$$\xi\colon \Sigma^4 M^R \longrightarrow M^R,\ c\colon M^R \longrightarrow M^C \text{ and } r\colon M^C \longrightarrow M^R$$

which satisfy the relations (1.7) i), iii) and moreover

(2.1) i) $\eta_R B_R = B_R \eta_R,\ c\eta_R = 0,\ \eta_R r = 0,$

ii) $cB_R = B_C^4 c,\ rB_C^4 = B_R r,\ \psi_C^{-1} c = c,\ r\psi_C^{-1} = r$ and

iii) $cr = 1 + \psi_C^{-1},\ rc = 2,\ rB_C c = \eta_R^2,\ rB_C^2 c = \xi,\ rB_C^3 c = 0.$

An object $M = \{M^C, M^R\}$ of CR is called *CR-acyclic* when the two sequences

(2.2) i) $\cdots \longrightarrow \Sigma^1 M^R \xrightarrow{\eta_R} M^R \xrightarrow{c} M^C \xrightarrow{rB_C^{-1}} \Sigma^2 M^R \xrightarrow{\eta_R} \Sigma^1 M^R \longrightarrow \cdots$

ii) $\cdots \longrightarrow \eta_R \Sigma^2 M^R \xrightarrow{\eta_R} \eta_R \Sigma^1 M_R \xrightarrow{\eta_R} \eta_R M^R \longrightarrow \cdots$

are exact. Obviously $K_*^{CR}P \wedge X$ is CR-acyclic for any CW-spectrum X, and $K_*^{CR}SG$ is also CR-acyclic for any abelian group G. Let $0 \to M \to N \to L \to 0$ be a short exact sequence in CR. If M satisfies the condition (2.2) i), then the sequence $0 \to \eta_R M^R \to \eta_R N^R \to \eta_R L^R \to 0$ becomes exact, too.

An object of CT is a pair $M = \{M^C, M^T\}$ consisting of a KU_*-and KT_*-module M^C and M^T equipped with the operations below. Thus M^C and M^T are united by the following KO_*-module maps:

$$B_C\colon \Sigma^2 M^C \xrightarrow{\simeq} M^C,\ \psi_C^{-1}\colon M^C \xrightarrow{\simeq} M^C,\ B_T\colon \Sigma^4 M^T \xrightarrow{\simeq} M^T,\ \eta_T\colon \Sigma^1 M^T \longrightarrow M^T,$$

$$\omega\ \colon \Sigma^3 M^T \longrightarrow M^T,\ \psi_T^{-1}\colon M^T \xrightarrow{\simeq} M^T,\ \zeta\colon M^T \longrightarrow M^C,\ \gamma\colon M^C \longrightarrow \Sigma^1 M^T \text{ and}$$

$$\varepsilon\tau\ \colon \Sigma^1 M^T \longrightarrow M^T$$

which satisfy the relations (1.7) ii), iii), iv), vii), ix) and moreover

(2.3) i) $\eta_T B_T = B_T \eta_T,\ \eta_T \psi_T^{-1} = \psi_T^{-1} \eta_T = \eta_T,\ \zeta\eta_T = 0,\ \eta_T \gamma = 0,$

$\eta_T \varepsilon\tau = \varepsilon\tau\eta_T = \varepsilon\tau\varepsilon\tau,$

ii) $\varepsilon\tau B_T^2 = B_T^2 \varepsilon\tau,\ \psi_T^{-1}\varepsilon\tau = -\varepsilon\tau\psi_T^{-1} = \varepsilon\tau,\ B_T \varepsilon\tau B_T^{-1} = \varepsilon\tau + \eta_T$ and

iii) $\zeta\varepsilon\tau\gamma = 1 + \psi_C^{-1},\ \varepsilon\tau\zeta\gamma = 1 + \psi_T^{-1},\ \gamma\zeta\varepsilon\tau = 1 - \psi_T^{-1}.$

An object $M = \{M^C, M^T\}$ of CT is called *CT-acyclic* when the two sequences

$$(2.4)\quad \text{i)}\quad \cdots \longrightarrow M^T \xrightarrow{\zeta} M^C \xrightarrow{B_C^{-1}(1-\psi_C^{-1})} \Sigma^2 M^C \xrightarrow{\gamma B_C} \Sigma^1 M^T \xrightarrow{\zeta} \Sigma^1 M^C \longrightarrow \cdots$$

$$\text{ii)}\quad \cdots \longrightarrow \eta_T \Sigma^2 M^T \xrightarrow{\varepsilon\tau} \eta_T \Sigma^1 M^T \xrightarrow{\varepsilon\tau} \eta_T M^T \longrightarrow \cdots$$

are exact. Obviously $K_*^{CT}P \wedge X$ is CT-acyclic for any CW-spectrum X, and $K_*^{CT}SH$ and $K_*^{CT}Q \wedge SG$ are CT-acyclic for any abelian groups H and G with H uniquely 2-divisible. Let $0 \to M \to N \to L \to 0$ be a short exact sequence in CT. If both M and N satisfy the condition (2.4) i), then the sequence $0 \to \eta_T M^T \to \eta_T N^T \to \eta_T L^T \to 0$ becomes exact. We moreover note that $\gamma\zeta(\mathrm{Ker}\,\eta_T) = 2\gamma M^C$ and $\gamma\zeta(\mathrm{Ker}\,\eta_T * Z/2) \subset \eta_T M^T$ when an object $M = \{M^C, M^T\}$ of CT satisfies the condition (2.4) i).

An object of RT is a pair $M = \{M^R, M^T\}$ consisting of a KO_*- and KT_*-module M^R and M^T equipped with the operations below. Thus M^R and M^T are united by the following KO_*-module maps:

$$B_R\colon \Sigma^8 M^R \xrightarrow{\simeq} M^R,\ \eta_R\colon \Sigma^1 M^R \longrightarrow M^R,\ \xi\colon \Sigma^4 M^R \longrightarrow M^R,\ B_T\colon \Sigma^4 M^T \xrightarrow{\simeq} M^T,$$

$$\eta_T : \Sigma^1 M^T \longrightarrow M^T,\ \omega\colon \Sigma^3 M^T \longrightarrow M^T,\ \psi_T^{-1}\colon M^T \xrightarrow{\simeq} M^T,\ \varepsilon\colon M^R \longrightarrow M^T,$$

$$\tau\ :\Sigma^1 M^T \longrightarrow M^R \quad\text{and}\quad \gamma\zeta\colon M^T \longrightarrow \Sigma^1 M^T$$

which satisfy the relations (1.7) i), ii), iv), vi), viii) and moreover

$$(2.5)\quad \text{i)}\quad \eta_R B_R = B_R \eta_R,\ \eta_T B_T = B_T \eta_T,\ \eta_T \psi_T^{-1} = \psi_T^{-1}\eta_T = \eta_T,\ \varepsilon\eta_R = \eta_T\varepsilon,$$

$$\tau\eta_T = \eta_R\tau,\ \gamma\zeta\eta_T = 0 = \eta_T\gamma\zeta,$$

$$\text{ii)}\quad \gamma\zeta B_T = B_T\gamma\zeta = \omega,\ \psi_T^{-1}\gamma\zeta = -\gamma\zeta\psi_T^{-1} = -\gamma\zeta,\ \gamma\zeta\gamma\zeta = 0 \quad\text{and}$$

$$\text{iii)}\quad \tau\gamma\zeta\varepsilon = 2,\ \varepsilon\tau\gamma\zeta = 1 + \psi_T^{-1},\ \gamma\zeta\varepsilon\tau = 1 - \psi_T^{-1}.$$

An object $M = \{M^R, M^T\}$ of RT is called *RT_F-acyclic* when the two sequences

$$(2.6)_F\quad \text{i)}\quad \cdots \longrightarrow \Sigma^2 M^R \xrightarrow{\eta_R^2} M^R \xrightarrow{\varepsilon} M^T \xrightarrow{\tau B_T^{-1}} \Sigma^3 M^R \xrightarrow{\eta_R^2} \Sigma^1 M^R \longrightarrow \cdots$$

$$\text{ii)}\quad \cdots \longrightarrow M^T/\mathrm{Im}\,\eta_T \xrightarrow{\gamma\zeta} \Sigma^1 M^T/\mathrm{Im}\,\eta_T \xrightarrow{\gamma\zeta} \Sigma^2 M^T/\mathrm{Im}\,\eta_T \longrightarrow \cdots$$

are exact, and in addition

iii) $\gamma\zeta(\mathrm{Ker}\,\eta_T) \subset 2M^T$.

For any 2-torsion free abelian group G, $K_*^{RT}SG$ and $K_*^{RT}Q \wedge SG$ are RT_F-acyclic.

An object $M = \{M^R, M^T\}$ of RT is called *RT_I-acyclic* when the two sequences

(2.6)$_I$ i) $\cdots \longrightarrow \Sigma^2 M^R \xrightarrow{\eta_R^2} M^R \xrightarrow{\varepsilon} M^T \xrightarrow{\tau B_T^{-1}} \Sigma^3 M^R \xrightarrow{\eta_R^2} \Sigma^1 M^R \longrightarrow \cdots$

ii) $\cdots \longrightarrow \mathrm{Ker}\,\eta_T \xrightarrow{\gamma\zeta} \mathrm{Ker}\,\eta_T \xrightarrow{\gamma\zeta} \mathrm{Ker}\,\eta_T \longrightarrow \cdots$

are exact, and in addition

iii) $\gamma\zeta(\mathrm{Ker}\,\eta_T * Z/2) \subset \eta_T M^T$.

For any 2-divisible abelian group G, $K_*^{RT}SG$ and $K_*^{RT}Q \wedge SG$ are RT_I-acyclic.

2.2. To deal with only the much simpler K-homology KU_*X, KO_*X or KT_*X, we next introduce new abelian categories C, R and T.

An object of C is a KU_*-module M^C equipped with operations

$$B_C\colon \Sigma^2 M^C \xrightarrow{\simeq} M^C \quad \text{and} \quad \psi_C^{-1}\colon M^C \xrightarrow{\simeq} M^C$$

satisfying the relation (1.7) iii) (see [9, 4.1]). An object M^C of C is called C-*acyclic* (or *Inv-acyclic* [9, 3.5]) when the sequence

$$(2.7) \qquad \cdots \longrightarrow M^C \xrightarrow{1+\psi_C^{-1}} M^C \xrightarrow{1-\psi_C^{-1}} M^C \xrightarrow{1+\psi_C^{-1}} M^C \longrightarrow \cdots$$

is exact. Obviously $KU_*P \wedge X$ is C-acyclic for any CW-spectrum X.

An object of R is just a KO_*-module M^R whose operations

$$B_R\colon \Sigma^8 M^R \xrightarrow{\simeq} M^R,\ \eta_R\colon \Sigma^1 M^R \longrightarrow M^R \quad \text{and} \quad \xi\colon \Sigma^4 M^R \longrightarrow M^R$$

satisfy the relation (1.7) i).

An object of T is a KT_*-module M^T equipped with the following operations

$$B_T\colon \Sigma^4 M^T \xrightarrow{\simeq} M^T,\ \eta_T\colon \Sigma^1 M^T \longrightarrow M^T,\ \omega\colon \Sigma^3 M^T \longrightarrow M^T,\ \psi_T^{-1}\colon M^T \xrightarrow{\simeq} M^T,$$

$$\varepsilon\tau\colon \Sigma^1 M^T \longrightarrow M^T \quad \text{and} \quad \gamma\zeta\colon M^T \longrightarrow \Sigma^1 M^T$$

which satisfy the relations (1.7) ii), iv) and moreover

(2.8) i) $\eta_T B_T = B_T \eta_T,\ \eta_T \psi_T^{-1} = \psi_T^{-1}\eta_T = \eta_T,\ \eta_T \varepsilon\tau = \varepsilon\tau\eta_T = \varepsilon\tau\varepsilon\tau,$

$\eta_T\gamma\zeta = 0 = \gamma\zeta\eta_T.$

ii) $\varepsilon\tau B_T^2 = B_T^2\varepsilon\tau,\ \psi_T^{-1}\varepsilon\tau = -\varepsilon\tau\psi_T^{-1} = \varepsilon\tau,\ B_T\varepsilon\tau B_T^{-1} = \varepsilon\tau + \eta_T,$

iii) $\gamma\zeta B_T = B_T\gamma\zeta = \omega,\ \psi_T^{-1}\gamma\zeta = -\gamma\zeta\psi_T^{-1} = -\gamma\zeta,\ \gamma\zeta\gamma\zeta = 0$ and

iv) $\varepsilon\tau\gamma\zeta = 1 + \psi_T^{-1},\ \gamma\zeta\varepsilon\tau = 1 - \psi_T^{-1}.$

An object M^T of T is called *T_F-acyclic* when the two sequences

$$(2.9)_F \quad \text{i)} \quad \cdots \longrightarrow \eta_T\Sigma^2 M^T \xrightarrow{\varepsilon\tau} \eta_T\Sigma^1 M^T \xrightarrow{\varepsilon\tau} \eta_T M^T \longrightarrow \cdots$$

$$\text{ii)} \quad \cdots \longrightarrow M^T/\mathrm{Im}\eta_T \xrightarrow{\gamma\zeta} \Sigma^1 M^T/\mathrm{Im}\eta_T \xrightarrow{\gamma\zeta} \Sigma^2 M^T/\mathrm{Im}\eta_T \longrightarrow \cdots$$

are exact, and in addition

iii) $\gamma\zeta(\mathrm{Ker}\eta_T) \subset 2M^T$.

For any 2-torsion free abelian group G, $KT_*Q \wedge SG$ is T_F-acyclic.

An object M^T of T is called *T_I-acyclic* when the two sequences

$$(2.9)_I \quad \text{i)} \quad \cdots \longrightarrow \eta_T\Sigma^2 M^T \xrightarrow{\varepsilon\tau} \eta_T\Sigma^1 M^T \xrightarrow{\varepsilon\tau} \eta_T M^T \longrightarrow \cdots$$

$$\text{ii)} \quad \cdots \longrightarrow \mathrm{Ker}\eta_T \xrightarrow{\gamma\zeta} \mathrm{Ker}\eta_T \xrightarrow{\gamma\zeta} \mathrm{Ker}\eta_T \longrightarrow \cdots$$

are exact, and in addition

iii) $\gamma\zeta(\mathrm{Ker}\eta_T * Z/2) \subset \eta_T M^T$.

For any 2-divisible abelian group G, $KT_*Q \wedge SG$ is T_I-acyclic.

2.3. Let $\mathscr{C}$ denote one of the abelian categories CR, CT, RT, C, R and T. As in the CRT case a KO-module spectrum W is said to be *$\pi_*^{\mathscr{C}}$-free* if it has the following form: $W = KU \wedge SA \vee KO \wedge SB$, $KU \wedge SA \vee KT \wedge SC$, $KO \wedge SB \vee KT \wedge SC$, $KU \wedge SA$, $KO \wedge SB$ or $KT \wedge SC$ according as $\mathscr{C} = CR$, CT, RT, C, R or T, where $A = \{A_i\}_{0 \le i \le 1}$, $B = \{B_j\}_{0 \le j \le 7}$ and $C = \{C_k\}_{0 \le k \le 3}$ are all graded free. Dually a KO-module spectrum W is said to be *$\pi_*^{\mathscr{C}}$-cofree* if it has the following form: $W = KU \wedge SA \vee KO \wedge SB$, $KU \wedge SA \vee KT \wedge SC \vee KO \wedge SD$, $KO \wedge SB \vee KT \wedge SC$, $KU \wedge SA \vee KO \wedge SD$, $KO \wedge SB$ or $KT \wedge SC \vee KO \wedge SD$ according as $\mathscr{C} = CR$, CT, RT, C, R or T, where $A = \{A_i\}_{0 \le i \le 1}$ and $C = \{C_k\}_{0 \le k \le 3}$ are graded divisible 2-torsion, $B = \{B_j\}_{0 \le j \le 7}$ is graded divisible and $D = \{D_j\}_{0 \le j \le 3}$ is graded divisible 2-torsion free.

A *free* object of $\mathscr{C}$ is isomorphic in $\mathscr{C}$ to a certain homotopy $\pi_*^{\mathscr{C}} W$ with W $\pi_*^{\mathscr{C}}$-free, and a *cofree* object of $\mathscr{C}$ is isomorphic to $\pi_*^{\mathscr{C}} W$ with W $\pi_*^{\mathscr{C}}$-cofree.

We shall later use the following result similar to [9, Proposition 3.11] for a CRT-acyclic object M.

LEMMA 2.1. i) *Let $M = \{M^C, M^R\}$ be an object of CR satisfying the condition* (2.2) i). *Then $\eta_R M_R = 0$ if and only if M^C is C-acyclic.*

ii) *Let $M = \{M^C, M^T\}$ be an object of CT satisfying the condition* (2.4) i). *Then $\eta_T M^T = 0$ if and only if M^C is C-acyclic.*

iii) *Let* $M=\{M^R, M^T\}$ *be an object of* RT *satisfying the condition* $(2.6)_F$ i). *Then* $\eta_R M^R=0$ *if and only if* $\eta_R^2 M^R=0$ *and* $\eta_T \varepsilon\tau M^T=0$.

PROOF. i) The "only if" part is easy. To prove the converse we take an arbitrary elment $x\in M^R$. Then it is expressed as $x=ry+\eta_R z$ for some $y\in M^C$ and $z\in M^R$. So we see that $\eta_R x=\eta_R^2 z$ and $\eta_R^2 x=0$. This means that $\eta_R x=0$ as desired,

ii) and iii) are easily shown by routine arguments.

Let $M=\{M^C, M^R, M^T\}$ be an acyclic object of CRT such that anyone of M^C, M^R and M^T is uniquely 2-divisible. Then the others of them are uniquely 2-divisible and hence $\eta_R=0$ and $\eta_T=0$. So we obtain natural decompositions $\mathrm{M}^R\cong(M^T)^+\cong(M^C)^+$, $M^C\cong(M^C)^+\oplus(M^C)^-\cong M^R\oplus\Sigma^2M^R$ and $M^T\cong(M^T)^+\oplus(M^T)^-\cong M^R\oplus\Sigma^{-1}M^R$ where $(M_C)^\pm=\mathrm{Ker}(1\pm\psi_C^{-1})\subset M^C$ and $(M^T)^\pm=\mathrm{Ker}(1\pm\psi_T^{-1})\subset M^T$. Moreover there exists uniquely a periodicity operation $B_R^{1/2}\colon\Sigma^4M^R\to M^R$ satisfying $\varepsilon B_R^{1/2}=B_T\varepsilon$, $B_R^{1/2}\tau=\tau B_T$ and $B_R^{1/2}B_R^{1/2}=B_R$. Replacing M^C and M^T by $M^R\oplus\Sigma^2M^R$ and $M^R\oplus\Sigma^{-1}M^R$ respectively we can rewrite the operations of M in (1.6) as follows (cf. [9, 4.2]):

$$B_C(x, y)=(By, x),\ B_R(z)=B^2(z),\ B_T(u, w)=(Bu, Bw),$$

$$\psi_C^{-1}(x, y)=(x, -y),\ \psi_T^{-1}(u, w)=(u, -w),\ \eta_R(z)=0,\ \eta_T(u, w)=0,$$

$$\varepsilon(z)=(z, 0),\ \zeta(u, w)=(u, 0),\ \gamma(x, y)=(0, 2x)\quad\text{and}\quad\tau(u, w)=w$$

in which $(x, y)\in M_*^R\oplus M_{*-2}^R$, $z\in M_*^R$, $(u, w)\in M_*^R\oplus M_{*+1}^R$ and the periodicity operation $B_R^{1/2}$ is abbreviated as B.

This implies easily

LEMMA 2.2. *Let* $\mathscr{C}$ *be one of the abelian categories* CR, CT, RT, C, R *and* T, *and* $M=\{M^H\}$ *be a* $\mathscr{C}$*-acyclic object such that anyone of the entries* M^H *of* M *is uniquely* 2*-divisible. Then* M *is extended to a certain* CRT*-acyclic object* $\rho M=\{M^C, M^R, M^T\}$.

By combining the above lemma with Lemma 1.3 we can show

COROLLARY 2.3. *Let* $\mathscr{C}$ *and* M *be ones stated in the above lemma. Then* M *is isomorphic in* $\mathscr{C}$ *to the K-homology* $K_*^{\mathscr{C}}SD$ *where* $D=\{M_j^R\}_{0\le j\le 3}$.

§3. Projective and injective objects in $\mathscr{C}$

3.1. We now give criteria for projective objects in the abelian categories CR, CT and RT introduced in the previous section corresponding to Theorem

1.2 for the abelian category CRT (see [9, Proposition 4.8] in the CR case).

THEOREM 3.1. (1) *Let $\mathscr{C}$ be the abelian category CR or CT. For an object M of $\mathscr{C}$ the following conditions are equivalent:*

i) *M is projective in $\mathscr{C}$,*

ii) *M is $\mathscr{C}$-acyclic with M^C free, and*

iii) *M is free in $\mathscr{C}$.*

(2) *For an object M of the abelian category RT the following conditions are equivalent:*

i) *M is projective in RT,*

ii) *M is RT_F-acyclic with $M^R/\mathrm{Im}\,\eta_R$ and $M^T/\mathrm{Im}\,\eta_T$ free, and*

iii) *M is free in RT.*

PROOF. It is straightforward to prove (1) by using the method developed in [9, Proof of Theorem 3.2]. In order to prove (2) we mimic the Bousfield's method with a minor device. It is sufficient to show only the implication ii) $\to$ iii).

Decompose the free abelian group $M_0^T/\mathrm{Im}\,\eta_T$ as $G \oplus \psi_T^{-1} G \oplus i^+ H \oplus i^- I$ where $G \oplus \psi_T^{-1} G$, $i^+ H$ and $i^- I$ respectively denote $G \oplus G$ with ψ_T^{-1} interchanging summands, H with $\psi_T^{-1} = 1$ and I with $\psi_T^{-1} = -1$. The homomorphism $\eta_R^2\colon M_0^R \to M_2^R$ is factorized through $(M_0^T)^+/\mathrm{Im}\,\eta_T$ as

$$M_0^R \xrightarrow{\varepsilon} (M_0^T)^+/\mathrm{Im}\,\eta_T \xrightarrow{\eta_R \tau} \eta_R \tau (M_0^T)^+ \subset M_2^R$$

where $(M_0^T)^+/\mathrm{Im}\,\eta_T \cong \Delta_+ G \oplus i^+ H$ with $\Delta_+ G = \{(g, \psi_T^{-1} g) \in G \oplus \psi_T^{-1} G\}$. Choose a decomposition $i^+ H \cong B \oplus C \oplus D$ so that $\eta_R \tau B \cong B \otimes Z/2 \cong \eta_R \tau (M_0^T)^+/\eta_R^2 M_0^R$, $\eta_R \tau C \cong C \otimes Z/2 \cong \eta_R^2 M_0^R$ and $\eta_R \tau D = 0$. Setting $C = \bigoplus_\gamma Z\{c_\gamma\}$, we get an element $m_\gamma \in M_0^R$ such that $\varepsilon m_\gamma - c_\gamma \in \Delta_+ G \oplus D$ for each γ as $2(M_0^T)^+ \subset \varepsilon M_0^R$. Since $\{\eta_R^2 m_\gamma\}$ forms a basis of $\eta_R^2 M_0^R$, we may regard as $C = \bigoplus_\gamma Z\{\varepsilon m_\gamma\}$. Consider the homomorphism $f\colon K_*^{RT} SC \to M$ defined by $f(b_{\gamma,R}) = m_\gamma$ for each γ where $SC = \bigvee_\gamma \Sigma_\gamma^0$ and $b_{\gamma,R} \in KO_0 \Sigma_\gamma^0$ denotes the standard generator. As is easily seen, f is a monomorphism. Denote by $\bar{M} = \{\bar{M}^R, \bar{M}^T\}$ the cokernel of the map f. Then the short exact sequences $0 \to KT_k SC/\mathrm{Im}\,\eta_T \to M_k^T/\mathrm{Im}\,\eta_T \to \bar{M}_k^T/\mathrm{Im}\,\eta_T \to 0$ and $0 \to KO_j SC/\mathrm{Im}\,\eta_R \to M_j^R/\mathrm{Im}\,\eta_R \to \bar{M}_j^R/\mathrm{Im}\,\eta_R \to 0$ are evidently split except $k \equiv 3 \bmod 4$ and $j \equiv 4 \bmod 8$. The homomorphism $\gamma\zeta\colon M_0^T/\mathrm{Im}\,\eta_T \to M_{-1}^T/\mathrm{Im}\,\eta_T$ restricted to $G \oplus i^+ H$ gives rise to an isomor-phism $\gamma\zeta\colon G \oplus i^+ H \to \gamma\zeta(M_0^T/\mathrm{Im}\,\eta_T)$. On the other hand, $\gamma\zeta(M_0^T/\mathrm{Im}\,\eta_T)$ is a direct summand of $M_{-1}^T/\mathrm{Im}\,\eta_T$ under the condition $(2.6)_F$ ii) and the freeness of $M^T/\mathrm{Im}\,\eta_T$. Hence the above former sequence becomes split even if $k \equiv 3 \bmod 4$. To observe that the lattter sequence is split when

$j \equiv 4 \bmod 8$, we next show that $\varepsilon: \bar{M}_4^R \to \bar{M}_4^T$ reduces to a monomorphism $\varepsilon: \bar{M}_4^R/\mathrm{Im}\eta_R \to \bar{M}_4^T/\mathrm{Im}\eta_T$. Take an arbitrary element $\bar{x} \in \bar{M}_4^R$ with $\varepsilon\bar{x} \in \eta_T \bar{M}_3^T$, and then choose an element $x \in M_4^R$ projecting to $\bar{x} \in \bar{M}_4^R$. The homomorphism $\varepsilon: KO_4SC \to KT_4SC$ is just multiplication by 2 on C and $\varepsilon: M_4^R/\mathrm{Im}\eta_R \to M_4^T/\mathrm{Im}\eta_T$ is a monomorphism. So there exists an element $y \in KO_0SC$ with $2x - f^R(\xi y) \in \eta_R M_3^R$. Then it follows immediately that $\varepsilon x - B_T\varepsilon f^R(y) \in \eta_T M_3^T$ and hence $\eta_R^2 f^R(y) = 0$. Taking an element $y' \in KO_0SC$ with $2y' = y$, we see easily that $x - f^R(\xi y') \in \eta_R M_3^R$, thus $\bar{x} \in \eta_R \bar{M}_3^R$ as desired. Consequently we obtain a RT_F-acyclic object $\bar{M} = \{\bar{M}^R, \bar{M}^T\}$ such that $\bar{M}^R/\mathrm{Im}\eta_R$ and $\bar{M}^T/\mathrm{Im}\eta_T$ are free and $\eta_R^2 \bar{M}_0^R = 0$. Repeat this construction in successive dimensions to give finally a RT_F-acyclic object $N = \{N^R, N^T\}$ with $N^R/\mathrm{Im}\eta_R$ and $N^T/\mathrm{Im}\eta_T$ free, and $\eta_R^2 N^R = 0$.

Next decompose the free abelian group $N_0^T/\mathrm{Im}\eta_T$ as $G \oplus \psi_T^{-1} G \oplus i^+ H \oplus i^- I$. The homomorphism $\eta_T\varepsilon\tau: N_0^T/\mathrm{Im}\eta_T \to N_2^T$ restricted to G gives rise to an isomorphism $\eta_T\varepsilon\tau: G \otimes Z/2 \to \eta_T\varepsilon\tau(N_0^T/\mathrm{Im}\eta_T)$ under the conditions $(2.6)_F$ ii) and iii). Set $G = \bigoplus_\sigma Z\{g_\sigma\}$ and consider the homomorphism $h: K_{*+3}^{RT}Q \wedge SG \to N$ defined by $h(b_{\sigma,T}) = g_\sigma$ for each σ where $SG = \bigvee_\sigma \Sigma_\sigma^0$ and $b_{\sigma,T} \in KT_3Q \wedge \Sigma_\sigma^0$ denotes the standard generator as given in (1.9) iii). It is evident that h is a monomorphism. Denote by $j: N_0^T/\mathrm{Im}\eta_T \to G \oplus \psi_T^{-1} G$ the canonical projection, which gives a left inverse of $h_0^T: KT_3Q \wedge SG \cong G \oplus \psi_T^{-1} G \to N_0^T/\mathrm{Im}\eta_T$. The compositions $\gamma\zeta\pi j\varepsilon\tau: N_{-1}^T/\mathrm{Im}\eta_T \to KT_2Q \wedge SG \cong G$ and $\varepsilon\tau\pi j\gamma\zeta: N_1^T/\mathrm{Im}\eta_T \to KT_4Q \wedge SG/\mathrm{Im}\eta_T \cong G$ give left inverses of h_{-1}^T and h_1^T respectively, where $\pi: G \oplus \psi_T^{-1} G \to G$ denotes the projection onto the first factor. Denote by $\bar{N} = \{\bar{N}^R, \bar{N}^T\}$ the cokernel of the map h. Then the short exact sequences $0 \to KT_{*+3}Q \wedge SG/\mathrm{Im}\eta_T \to N^T/\mathrm{Im}\eta_T \to \bar{N}^T/\mathrm{Im}\eta_T \to 0$ and hence $0 \to KO_{*+3}Q \wedge SG/\mathrm{Im}\eta_R \to N^R/\mathrm{Im}\eta_R \to \bar{N}^R/\mathrm{Im}\eta_R \to 0$ are split. Consequently we obtain a RT_F-acyclic object $\bar{N} = \{\bar{N}^R, \bar{N}^T\}$ such that $\bar{N}^R/\mathrm{Im}\eta_R$ and $\bar{N}^T/\mathrm{Im}\eta_T$ are free, $\bar{N}_0^T/\mathrm{Im}\eta_T \cong (\bar{N}_0^T/\mathrm{Im}\eta_T)^+ \oplus (\bar{N}_0^T/\mathrm{Im}\eta_T)^-$, $\eta_R^2 \bar{N}^R = 0$ and $\eta_T\varepsilon\tau\bar{N}_0^T = 0$. Repeat this construction in successive dimensions to give finally a RT_F-acylic object $L = \{L^R, L^T\}$ with L^R and L^T free and $L^T \cong (L^T)^+ \oplus (L^T)^-$. Here Lemma 2.1 iii) is needed to observe that $\eta_R L^R = 0$ and hence both L^R and L^T are free. Under the condition $(2.6)_F$ an arbitrary element $x \in L^T$ is expressed as a sum $\varepsilon\tau y + \gamma\zeta z$ for some $y, z \in L^T$. Then a routine computation shows that L^T is always 2-divisible and hence it must be trivial. Thus the original object M is free in RT.

3.2. As a dual of Theorem 3.1 we next give criteria for injective objects in the abelian categories CR, CT *and* RT corresponding to Theorem 1.4 for the abelian category CRT (see [9, Proposition 4.9] in the CR case).

THEOREM 3.2. (1) *Let $\mathscr{C}$ be the abelian category CR or CT. For an object M of $\mathscr{C}$ the following conditions are equivalent*:

i) *M is injective in $\mathscr{C}$,*

ii) *M is $\mathscr{C}$-acyclic with M^C* divisible, and

iii) *M is cofree in $\mathscr{C}$.*

(2) *For an object M of the abelian category RT the following conditions are equivalent*:

i) *M is injective in RT,*

ii) *M is RT_I-acyclic with* $\mathrm{Ker}\,\eta_R$ *and* $\mathrm{Ker}\,\eta_T$ *divisible , and*

iii) *M is cofree in RT.*

PROOF. It is straightforward to prove (1) by using the method in [9, Proof of Theorem 3.3]. In order to prove (2) we mimic the Bousfield's method with a minor device as in the proof of Theorem 3.1 (2). It is sufficient to show only the implication ii) $\to$ iii).

Let D be a divisible 2-torsion group with $\eta_R^2 M_1^R \cong D*Z/2$. Choose an epimorphism $\alpha'\colon (\mathrm{Ker}\,\eta_R)_3 \to D$ extending the identity on $\eta_R^2 M_1^R$ when $\eta_R^2 M_1^R$ is identified with $D*Z/2$, and then extend it to an epimorphism $\alpha\colon M_3^R \to D$. By means of the RT-version of Lemma 1.3 we get a homomorphism $f\colon M \to K_{*+1}^{RT} SD$ such that $f_3^R\colon M_3^R \to KO_4 SD \cong D$ is just the above α. It is easily seen that f is an epimorphism. Denote by $\tilde{M} = \{\tilde{M}^R, \tilde{M}^T\}$ the kernel of the map f and by $\tilde{\eta}_R$ and $\tilde{\eta}_T$ the Hopf operations of $\tilde{M}_R$ and $\tilde{M}_T$ with emphasis. Notice that $f_j^R\colon \eta_R^{j-1} M_1^R \to KO_{j+1} SD*Z/2 \cong D*Z/2$ is an epimorphism when $j = 2$ or 3, and moreover $f_2^T\colon \varepsilon\eta_R M_1^R \to KT_3 SD*Z/2 \cong D*Z/2$ is an epimorphism. Then the short exact sequences $0 \to (\mathrm{Ker}\,\tilde{\eta}_R)_j \to (\mathrm{Ker}\,\eta_R)_j \to (\mathrm{Ker}\,\eta_R^D)_{j+1} \to 0$ and $0 \to (\mathrm{Ker}\,\tilde{\eta}_T)_k \to (\mathrm{Ker}\,\eta_T)_k \to (\mathrm{Ker}\,\eta_R^D)_{k+1} \to 0$ are evidently split except $j \equiv -1$ mod 8 and $k \equiv -1$ mod 4, in which the Hopf operations of $KO_* SD$ and $KT_* SD$ are written as η_R^D and η_T^D. For an arbitrary element $x \in \tilde{M}_{-1}^R$ with $\tilde{\eta}_R x = 0$ there exists an element $y \in M_2^T$ with $\eta_T y = 0$ such that x injects into $\tau B_T^{-1} y \in M_{-1}^T$ and $2f_2^T(y) = 0$ because $\tau B_T^{-1}\colon \mathrm{Ker}\,\eta_T \to \mathrm{Ker}\,\eta_R$ is an epimorphism and $\tau B_T^{-1}\colon KT_3 SD \to KO_0 SD$ is multiplication by 2 on D. We can now replace the old y by a new one satisfying $f_2^T(y) = 0$, by using the restricted epimorphism $f_2^T\colon \varepsilon\eta_R M_1^R \to KT_3 SD*Z/2$. Thus $\tau B_T^{-1}\colon (\mathrm{Ker}\,\tilde{\eta}_T)_2 \to (\mathrm{Ker}\,\tilde{\eta}_R)_{-1}$ is an epimorphism. This implies that the above remaining sequences are both split, too. Consequently we obtain a RT_I-acyclic object $\tilde{M} = \{\tilde{M}^R, \tilde{M}^T\}$ such that $\mathrm{Ker}\,\tilde{\eta}_R$ and $\mathrm{Ker}\,\tilde{\eta}_T$ are divisible and $\tilde{\eta}_R^2 M_1^R = 0$. Repeat this construction in successive dimensions to give finally a RT_I-acyclic object $N = \{N^R, N^T\}$ with $\mathrm{Ker}\,\eta_R$ and $\mathrm{Ker}\,\eta_T$ divisible, and $\eta_R^2 N^R = 0$.

Decompose the divisible abelian group $(\mathrm{Ker}\,\eta_T)_3 \subset N_3^T$ as $G \oplus \psi_T^{-1} G \oplus i^+ H \oplus i^- I$ with G 2-torsion. For arbitrary element $x \in N_1^T$ there exists an element $y \in N_3^T$ with $\eta_T y = 0$ and $\eta_T x = \gamma\zeta y$ under the condition $(2.6)_I$ ii).

Since $\mathrm{Ker}\eta_T$ is divisible, the old y can be replaced by a new one satisfying $2y=0$. Then the homomorphism $\eta_T\varepsilon\tau\colon N_1^T\to N_3^T$ is mapped onto $\varDelta_+G*Z/2$ under the condition $(2.6)_I$ iii), thus $\eta_T\varepsilon\tau N_1^T=\varDelta_+G*Z/2$. Choose an epimorphism $\beta\colon N_3^T\to G\oplus\psi_T^{-1}G$ extending the canonical projection $j\colon(\mathrm{Ker}\eta_T)_3\to G\oplus\psi_T^{-1}G$. By use of the RT-version of Lemma 1.3 we get a homomorphism $h\colon N\to K_{*+4}^{RT}Q\wedge SG$ such that $(\mu_Q\wedge 1)_*h_3^T=(\mu_Q\wedge 1)_*\beta\colon N_3^T\to KT_7SG\cong G$ where $\mu_Q\colon KT\wedge Q\to KT$ is the pairing appeared in (1.11). It is not difficult to see that h is an epimorphism because G has only 2-torsion. Denote by $\tilde N=\{\tilde N^R,\tilde N^T\}$ the kernel of the epimorphism h and by $\tilde\eta_R$ and $\tilde\eta_T$ the Hopf operations of $\tilde N^R$ and $\tilde N^T$. Since h induces epimorphisms $h\colon\mathrm{Ker}\eta_T*Z/2\to\mathrm{Ker}\eta_T^G*Z/2$ and $h\colon\mathrm{Ker}\eta_R*Z/2\to\mathrm{Ker}\eta_R^G*Z/2$, the short exact sequences $0\to\mathrm{Ker}\tilde\eta_T\to\mathrm{Ker}\eta_T\to\mathrm{Ker}\eta_T^G\to 0$ and $0\to\mathrm{Ker}\tilde\eta_R\to\mathrm{Ker}\eta_R\to\mathrm{Ker}\eta_R^G\to 0$ are split where the Hopf operations of $KT_*Q\wedge SG$ and $KO_*Q\wedge SG$ are written as η_T^G and η_R^G. Consequently we obtain a RT_I-acyclic object $\tilde N=\{\tilde N^R,\tilde N^T\}$ such that $\mathrm{Ker}\tilde\eta_R$ and $\mathrm{Ker}\tilde\eta_T$ are divisible, $(\mathrm{Ker}\tilde\eta_T)_3\cong(\mathrm{Ker}\tilde\eta_T)_3^+\oplus(\mathrm{Ker}\tilde\eta_T)_3^-$, $\tilde\eta_R^2\tilde N^R=0$ and $\tilde\eta_T\varepsilon\tau\tilde N_1^T=0$. Repeat this construction in successive dimensions to give finally a RT_I-acyclic object $L=\{L^R,L^T\}$ with L^R and L^T divisible and $L^T\cong(L^T)^+\oplus(L^T)^-$. However L^T and hence L^R must be 2-torsion free since an element $x\in L^T$ with $2x=0$ belongs to $(L^T)^+\cap(L^T)^-=\{0\}$. By applying the RT-version of Lemma 1.3 (or Corollary 2.3) we observe that the final object L is isomorphic in RT to the cofree object $K_*^{RT}SB$ with $B=\{L_j^R\}_{0\le j\le 3}$. Thus the original object M is cofree in RT.

3.3. We finally give the corresponding results for the abelian categories C, R and T to Theorems 1.2 and 1.4 in the CRT case (see [9, Propositions 3.6 and 3.8] in the C case).

THEOREM 3.3. (1) *For an object M^C of the abelian category C the following conditions are equivalent*:

i) *M^C is projective in C,*

ii) *M^C is C-acyclic and free, and*

iii) *M^C is free in C.*

(2) *For an object M^R of the abelian category R the following conditions are equivalent*:

i) *M^R is projective in R,*

ii) *M^R is projective as an KO_*-module, and*

iii) *M^R is free in R.*

(3) *For an object M^T of the abelian category T the following conditions are equivalent*:

i) *M^T is projective in T,*

ii) *M^T is T_F-acyclic with $M^T/\mathrm{Im}\eta_T$ free, and*

iii) *M^T is free in T.*

PROOF. (1) has already been shown in [9, Proposition 3.6], and (3) is immediately done by using the method in the proof of Theorem 3.1.

(2) It is sufficient to show only the implication ii) $\to$ iii). Set $M^C = KU_* \underset{KO_*}{\otimes} M^R$ and $M^T = KT_* \underset{KO_*}{\otimes} M^R$ for a projective KO_*-module M^R. Then the triple $M = \{M^C, M^R, M^T\}$ is viewed as a CRT-acyclic object with M^C free. According to Theorem 1.2 the object M is free in CRT. From Lemma 4.5 below it follows that the projective KO_*-module M^R is certainly isomorphic in R to some free object KO_*SB with $B = \{B_j\}_{0 \le j \le 7}$ free.

THEOREM 3.4. (1) *For an object M^C of the abelian category C the following conditions are equivalent*:

i) *M^C is injective in C,*
ii) *M^C is C-acyclic and divisible, and*
iii) *M^C is cofree in C.*

(2) *For an object M^R of the abelian category R the following conditions are equivalent*:

i) *M^R is injective in R,*
ii) *M^R is injective as a KO_*-module, and*
iii) *M^R is cofree in R.*

(3) *For an object M^T of the abelian category T the following conditions are equivalent*:

i) *M^T is injective in T,*
ii) *M^T is T_I-acyclic with* $\mathrm{Ker}\,\eta_T$ *divisible, and*
iii) *M^T is cofree in T.*

PROOF. (1) has already been shown in [9, Proposition 3.8], and (3) is immediately done by using the proof of Theorem 3.2.

(2) It is sufficient to show only the implication ii) $\to$ iii). Set $M^C = \mathrm{Hom}_{KO_*}(KU_*, M^R)$ and $M^T = \mathrm{Hom}_{KO_*}(KT_*, M^R)$ for an injective KO_*-module M^R. Then the triple $M = \{M^C, M^R, M^T\}$ is viewed as a CRT-acyclic object with M^C divisible. Theorem 1.4 combined with Lemma 4.4 below asserts that the KO_*-module M^R is certainly isomorphic in R to some cofree object KO_*SB with $B = \{B_j\}_{0 \le j \le 7}$ divisible.

§4. $K_*^{\mathscr{C}}$-injective spectra

4.1. Let us denote hereafter by $\mathscr{C}$ one of the abelian categories CRT, CR, CT, RT, C, R and T. Given CW-spectra X and Y a map $f: X \to Y$ is said to be *$K_*^{\mathscr{C}}$-monic* if it induces a monomorphism $f_*: K_*^{\mathscr{C}}X \to K_*^{\mathscr{C}}Y$. We call a CW-spectrum W *$K_*^{\mathscr{C}}$-injective* ([12] or [17]) if any $K_*^{\mathscr{C}}$-monic map $f: X \to Y$ induces an epimorphism $f^*: [Y, W] \to [X, W]$ (see [9, §9] for a different definition which is equivalent to ours). In the above definitions $K^{\mathscr{C}}$ may be regarded as the following KO-module spectrum: $K^{\mathscr{C}} = KU \vee KO \vee KT$, $KU \vee KO$, $KU \vee KT$, $KO \vee KT$, KU, KO or KT according as $\mathscr{C} = CRT$, CR, CT, RT, C, R or T. If a CW-spectrum W is $K_*^{\mathscr{C}}$-injective, then it is a quasi KO-module spectrum by [17, Lemma 1.4]. Here we mean by a *quasi* E-module spectrum W an E-module spectrum which is not necessarily associative for a fixed ring spectrum E. Thus the map $\iota \wedge 1: W \to E \wedge W$ admits a left inverse $\mu: E \wedge W \to W$ where $\iota: S \to E$ denotes the unit of E.

As a special case of [17, Proposition 1.6] we have the following result (see also [17, Proposition 3.7 ii)] when $\mathscr{C} = C$, R and T).

LEMMA 4.1. *A CW-spectrum W is $K_*^{\mathscr{C}}$-injective if and only if it is a retract of a certain extended KO-module spectrum $KO \wedge Y$ which is $\pi_*^{\mathscr{C}}$-cofree.*

Using [4, Theorem 2.8] and [3, Theorem 2.2] together we show

LEMMA 4.2. i) *Let K denote the periodic K-spectrum KU or KO. Then the smash product $K \wedge K$ is decomposed as a K-module spectrum into the wedge sum $\vee K$ of countable copies of K.*

ii) *The smash product $KT \wedge KT$* is decomposed as a KT-module spectrum into the wedge sum $(\vee KT) \vee (\vee \Sigma^3 KT)$ *of countable copies of KT and $\Sigma^3 KT$.*

PROOF. i) Recall [4, Theorem 2.8] that the product map $\nu: \pi_* KO \otimes KO_0 KO \to KO_* KO$ is an isomorphism. Set $G = KO_0 KO$, which is torsion free by [4, Proposition 2.1], and then choose a KO-module map $g: KO \wedge SG \to KO \wedge KO$ inducing the above isomorphism ν in the homotopy. Using the homotopy equivalence g the smash product $KU \wedge KU$ is written into the wedge sum $KU \wedge SG \vee \Sigma^2 KU \wedge SG$ as a KU-module spectrum. This implies that $G = KO_0 KO$ is exactly countable free because $KU_0 KU$ is so according to [3, Theorem 2.2]. The result is now easy.

ii) follows immediately from i) since the smash product $KT \wedge KT$ is written as a KT-module spectrum into the wedge sum $KT \wedge KO \vee \Sigma^3 KT \wedge KO$.

4.2. We now prove the following result as is expected.

LEMMA 4.3. *Let G be a divisible 2-torsion group. Then*

i) *$KU \wedge SG$ is never K_*^{RT}-injective,*

ii) *$KO \wedge SG$ is never K_*^{CT}-injective, and*

iii) *$KT \wedge SG$ is never K_*^{CR}-injective.*

PROOF. It is sufficient to show our result for $G = Z/2^\infty$.

i) Assume that $KU \wedge SZ/2^\infty$ is K_*^{RT}-injective. The map $\gamma \wedge 1: KU \wedge SZ/2 \to \Sigma^1 KT \wedge SZ/2$ is K_*^{RT}-monic since $\gamma_*: KU_* SZ/2 \to KT_{*-1} SZ/2$ is a monomorphism. So there exists a map $f: \Sigma^1 KT \wedge SZ/2 \to KU \wedge SZ/2^\infty$ such that the composite map $f(\gamma \wedge 1): KU \wedge SZ/2 \to KU \wedge SZ/2^\infty$ is the canonical map $1 \wedge i_2$ associated with the inclusion $Z/2 \to Z/2^\infty$. Obviously the map f induces a monomorphism $f_*: KT_1 SZ/2 \to KU_2 SZ/2^\infty$. However this is a contradiction because $\eta_{T*}: KT_0 SZ/2 \to KT_1 SZ/2$ is an isomorphism and $KU_1 SZ/2^\infty = 0$.

ii) Assume that $KO \wedge SZ/2^\infty$ is K_*^{CT}-injective. Since the map $\varepsilon \wedge 1: KO \wedge SZ/2 \to KT \wedge SZ/2$ is K_*^{CT}-monic, there exists a map $g: KT \wedge SZ/2 \to KO \wedge SZ/2^\infty$ such that the composite map $g(\varepsilon \wedge 1): KO \wedge SZ/2 \to KO \wedge SZ/2^\infty$ coincides with the canonical map $1 \wedge i_2$. Obviously the map g induces an isomorphism $g_*: KT_3 SZ/2 \to KO_3 SZ/2^\infty$. However this is a contradiction because $\eta_{R*}: KO_3 SZ/2^\infty \to KO_4 SZ/2^\infty$ is a monomorphism and $\eta_{T*}: KT_3 SZ/2 \to KT_4 SZ/2$ is trivial.

iii) Assume that $KT \wedge SZ/2^\infty$ is K_*^{CR}-injective. Since $(\varepsilon\tau B_C^{-1})_*: KU_{*+2} SZ/2 \to KT_* SZ/2$ is trivial, the map $(-\tau \wedge 1, \tau B_T^{-1} \wedge 1): KT \wedge SZ/2 \to (\Sigma^{-1} KO \vee \Sigma^3 KO) \wedge SZ/2$ becomes KO_*-monic by (1.3) iv). On the other hand, the map $1 \wedge i_P \wedge 1: KT \wedge SZ/2 \to KT \wedge P \wedge SZ/2$ is evidently KU_*-monic. So there exists a map $h: (KT \wedge P \vee \Sigma^{-1} KO \vee \Sigma^3 KO) \wedge SZ/2 \to KT \wedge SZ/2^\infty$ such that the composite map $h(1 \wedge i_P \wedge 1, -\tau \wedge 1, \tau B_T^{-1} \wedge 1): KT \wedge SZ/2 \to KT \wedge SZ/2^\infty$ coincides with the canonical map $1 \wedge i_2$. Obviously the map h induces an epimorphism $h_*: KT_2 P \wedge SZ/2 \oplus KO_3 SZ/2 \to KT_2 SZ/2^\infty$. How-ever the above h_* must be trivial because $\eta_{T*}: KT_2 P \wedge SZ/2 \to KT_3 P \wedge SZ/2$ is trivial, $\eta_{T*}: KT_2 SZ/2^\infty \to KT_3 SZ/2^\infty$ is a monomorphism, $\eta_{R*}: KO_2 SZ/2 \to KO_3 SZ/2$ is an epimorphism and $KT_1 SZ/2^\infty = 0$. This is a contradiction.

Using the argument developed in the proof of Lemma 4.3 ii) and iii) we can immediately show the following result, which was needed in the proof of Theorem 3.4 (2).

Lemma 4.4. *Let G be a divisible 2-torsion group. Then neither $KU_* SG$ nor $KT_* SG$ is injective as a KO_*-module, and $KU_* SG$ is not injective as a KT_*-module.*

Let G be a divisible 2-torsion free group. Then the extended KO-module spectrum $KO \wedge SG$ admits a unique KT-module structure such that the composite map $\gamma\zeta\varepsilon \wedge 1: KO \wedge SG \to \Sigma^1 KT \wedge SG$ is a KT-module map. Although the KT-module spectrum $KO \wedge SG$ is KT_*-injective, its homotopy $KO_* SG$ is never injective as a KT_*-module.

By a dual argument to the proof of Lemma 4.3 (or Lemma 4.4) we can easily show the following result, which was needed in the proof of Theorem 3.3 (2).

LEMMA 4.5. *Neither $\pi_* KU$ nor $\pi_* KT$ is projective as a KO_*-module, and $\pi_* KU$ is not projective as a KT_*-module.*

4.3. Combining Theorem 1.8 with Lemma 4.3 we obtain a stronger result than Lemma 4.1 when W is a KO-module spectrum.

PROPOSITION 4.6. *Let W be KO-module spectrum. Then W is $K_*^{\mathscr{C}}$-injective if and only if it is $\pi_*^{\mathscr{C}}$-cofree.*

PROOF. It is sufficient to show the "only if" part. If a KO-module spectrum W is $K_*^{\mathscr{C}}$-injective, then $\pi_* P \wedge W$ is divisible by means of Lemma 4.1. Then Theorem 1.8 asserts that the KO-module spectrum W is π_*^{CRT}-cofree. So we can easily observe that W is in fact $\pi_*^{\mathscr{C}}$-cofree by virtue of Lemma 4.3.

By using Lemmas 4.1 and 4.2 and Proposition 4.6 we show

THEOREM 4.7. *The following three conditions are equivalent:*
i) *W is a $K_*^{\mathscr{C}}$-injective spectrum,*
ii) *W is a quasi KO-module spectrum such that $KO \wedge W$ is $K_*^{\mathscr{C}}$-injective, and*
iii) *W is a quasi KO-module spectrum such that $KO \wedge W$ is $\pi_*^{\mathscr{C}}$-cofree.*

PROOF. If a KO-module spectrum Z is $\pi_*^{\mathscr{C}}$-cofree, then the smash product $KO \wedge Z$ is also $\pi_*^{\mathscr{C}}$-cofree by virtue of Lemma 4.2 i) for $K = KO$. Therefore the implication i) $\rightarrow$ ii) follows from Lemma 4.1. The inverse implication ii) $\rightarrow$ i) is immediate. On the other hand, the condition ii) is equivalent to iii) because of Proposition 4.6.

When $\mathscr{C} = R$, RT or CRT we have

LEMMA 4.8. *Let W* be a *KO-module spectrum. Then*
i) *W is KO_*-injective if and only if $\pi_* W$ is injective as a KO_*-module.*
ii) *W is K_*^{RT}-injective if and only if $\pi_* Q \wedge W$ is injective as a KT_*-module.*
iii) *W is K_*^{CRT}-injective if and only if $\pi_* P \wedge W$ is divisible.*

PROOF. The "only if" part is evident by Proposition 4.6.

The "if" part: i) When $\pi_* W$ is injective as a KO_*-module, there exists a natural isomorphism $\kappa_*^{KO}\colon [X, W] \rightarrow \mathrm{Hom}_{KO_*}(KO_* X, \pi_* W)$ for any CW-spectrum X. Hence the result is immediate.

ii) Similarly we see that $Q \wedge W$ is KT_*-injective when $\pi_* Q \wedge W$ is injective as a KT_*-module. From Lemma 4.1 it follows that $\pi_* P \wedge W$ is divisible because $P \wedge W$ is a retract of $P \wedge Q \wedge W$. Thus the KO-module spectrum W is π_*^{CRT}-cofree by means of Theorem 1.8. So it is in fact π_*^{RT}-cofree by virtue of Lemma 4.3 since $Q \wedge W$ is KT_*-injective.

iii) is immediate by use of Theorem 1.8.

Combining Theorem 4.7 with Lemma 4.8 we can easily show

THEOREM 4.9. *A CW-spectrum W is $K_*^{\mathscr{C}}$-injective if and only if it satisfies the following condition according as $\mathscr{C} = CRT$, CR, CT, RT, C, R* or *T:*

i) *W is a quasi KO-module spectrum such that KU_*W is divisible,*

ii) *W is a quasi KO-module spectrum such that KU_*W is divisible and $\eta_R KO_{*-1}W \xrightarrow{\eta_R} \eta_R KO_* W \xrightarrow{\eta_R} \eta_R KO_{*+1}W$ is exact,*

iii) *W is a quasi KT-module spectrum such that KU_*W is divisible,*

iv) *W is a quasi KO-module spectrum such that KT_*W is injective as a KT_*-module,*

v) *W is a quasi KU-module spectrum such that KU_*W is divisible,*

vi) *W is a quasi KO-module spectrum such that KO_*W is injective as a KO_*-module, or*

vii) *W is a quasi KT-module spectrum such that KT_*W is injective as a KT_*-module.*

§5. The abelian categories $A\mathscr{C}$ of KO_*KO-comodules

5.1. For any CW-spectrum X the united K-homology $K_*^{CRT}X = \{KU_*X, KO_*X, KT_*X\}$ admits a KO_*KO-comodule structure. Its comodule structure map $\psi_X = \{\psi_X^C, \psi_X^R, \psi_X^T\}: K_*^{CRT}X \to K_*^{CRT}KO \wedge X \xleftarrow{\cong} KO_*KO \underset{KO_*}{\otimes} K_*^{CRT}X$ is induced by the left unit map $1 \wedge \iota: KO \to KO \wedge KO$. In particular, ψ_S^R is the left unit map $\eta_L: \pi_*KO \to KO_*KO$ when $X = S$, the sphere spectrum. Recall that the product map $\nu: KO_0KO \otimes \pi_*KO \to KO_*KO$ is an isomorphism and KO_0KO is countable free. Choose a countable free basis $\{z_n\}$ of KO_0KO with $z_0 = \eta_L \iota$ and fix it. Set $\eta_L \xi = \sum_n z_n \otimes k_n\xi \in KO_0KO \otimes \pi_4KO$ for the generator $\xi \in \pi_4KO$ where k_n's are integers with $k_0 = 1$. Thus $\eta_L\xi = \sum_n k_n\eta_R\xi \cdot z_n \in KO_4KO$ where $\eta_R: \pi_*KO \to KO_*KO$ denotes the right unit map and "$\cdot$" stands for multiplication in KO_*KO.

In order to introduce the abelian categories $A\mathscr{C}$ consisting of objects M of $\mathscr{C}$ having a KO_*KO-comodule structure when $\mathscr{C} = CRT$, CR, CT, RT, C, R and T, we first represent the operations of $K_*^{CRT}KO \wedge X$ in terms of those of $K_*^{CRT}X$. The operations of $K_*^{CRT}KO \wedge X$ is written as $\tilde{f}$ in place of f in distinction from those of $K_*^{CRT}X$.

LEMMA 5.1. *When $K_*^{CRT}KO \wedge X$ is identified with $KO_*KO \underset{KO_*}{\otimes} K_*^{CRT}X$, the operations $\tilde{f}$ of $K_*^{CRT}KO \wedge X$ is expressed as follows:*

i) *$\tilde{f} = 1 \otimes f$ when $f = \eta_R, \eta_T, \psi_C^{-1}, \psi_T^{-1}, \omega, \varepsilon, \zeta, \tau B_T^{-1}$ or γB_C,*

ii) *$\tilde{f} = f \otimes 1$ when $f = B_R$ or ξ,*

iii) *$\tilde{B}_C = 1 \otimes B_C + \sum_{n \neq 0} z_n \otimes a_n(1 + \psi_C^{-1})B_C$, and*

iv) $\tilde{B}_T = \sum_n z_n \otimes k_n B_T$

where $\{z_n\}$ *is a fixed countable free basis of* KO_0KO, k_n *is the integer determined by the equality* $\eta_L\xi = \sum_n z_n \otimes k_n\xi$ *and* a_n *is a certain integer depending on the integer* k_n *and the multiplication of* KU.

PROOF. i) follows immediately since the operations given in i) are all represented by maps among Σ^n, $\Sigma^n P$ and $\Sigma^n Q$ smashed with KO. On the other hand, ii) is easy because the comodule structure map $\psi_X^R\colon KO_*X \to KO_*KO \wedge X \cong KO_*KO \underset{KO_*}{\otimes} KO_*X$ is a left KO_*-module homomorphism.

iii) Let $\mu\colon P \wedge P \to KO \wedge P$ denotes the map associated with the multiplication of KU. By a routine computation we can easily observe that $8\mu\colon P \wedge P \to KO \wedge P$ is at least decomposed as a sum $(\iota \wedge 1)\alpha + a(1 \wedge i_P)\xi(j_P \wedge j_P)$ for some map $\alpha\colon P \wedge P \to P$ and some integer a where $i_P\colon \Sigma^0 \to P$ and $j_p\colon P \to \Sigma^2$ denote the bottom cell inclusion and the top cell projection. In the above observation we may use the group structures $[P, \Sigma^n P]$ and $[P \wedge P, \Sigma^n P]$ calculated in [10, Lemmas 3.2 and 3.6] (and [11, Proposition 2.9]). Such a map α and an integer a are uniquely chosen. Recall that the periodicity generator $B_C \in \pi_2 KU$ is induced by a certain map $\beta \in \pi_2 P$ with $j_P\beta = 2$. Since $8\mu(\beta \wedge 1) = (\iota \wedge 1)\alpha(\beta \wedge 1) + 2a\xi \wedge i_P j_P$, we obtain that $8B_C(h) = (\alpha \wedge 1)_*(\beta \wedge 1 \wedge 1)_*h + 2a\xi\cdot(i_P j_P \wedge 1)_*h$ for each element $h \in KO_*P \wedge X$, in which $\xi\cdot$ stands for left multiplication by $\xi \in \pi_4 KO$. Hence it follows immediately that $8\tilde{B}_C(g \otimes h) = g \otimes 8B_C(h) + 2a(\eta_L\xi - \eta_R\xi)\cdot g \otimes (i_P j_P \wedge 1)_*h$ for each element $g \otimes h \in KO_*KO \underset{KO_*}{\otimes} KO_*P \wedge X$.

Take $g = z_0 = \iota \wedge \iota \in KO_0KO$ and $h = b_C \in KU_2P$ the standard generator given in (1.9) ii). Then the above equality implies that $\sum_{n \neq 0} \eta_R\xi\cdot z_n \otimes ak_n(1 + \psi_C^{-1})B_C^{-1}(b_C) = \sum_{n \neq 0} z_n \otimes 2ak_n(1 + \psi_C^{-1})B_C(b_C) \in KO_*KO \underset{KO_*}{\otimes} KU_*P$ is divided by 4 because $c\xi = 2B_C^2$. Therefore ak_n is divided by 2 except $n = 0$. Consequently the previous equality is rewritten as $\tilde{B}_C(g \otimes 8h) = g \otimes B_C(8h) + \sum_{n \neq 0} z_n\cdot g \otimes a_n(1 + \psi_C^{-1})B_C(8h)$ where $2a_n = ak_n$ for $n \neq 0$. This asserts that iii) is valid whenever KU_*X is 2-divisible. In fact iii) becomes always valid for any CW-spectrum X because there exists a monomorphism $f\colon K_*^{CRT}X \to K_*^{CRT}Y$ for some π_*^{CRT}-cofree $KO \wedge Y$.

iv) The multiplication of KT is induced by a certain pairing $m\colon Q \wedge Q \to Q$. Note that $\psi_S^T(B_T) = \sum_n z_n \otimes k_n B_T \in KO_0KO \otimes \pi_4 KT$ because $\varepsilon\xi = 2B_T$. Then it is easily seen that $\tilde{B}_T(g \otimes h) = \sum_n z_n\cdot g \otimes k_n B_T(h)$ for each element $g \otimes h \in KO_*KO \underset{KO_*}{\otimes} KO_*Q \wedge X$.

5.2. For an object $M=\{M^H\}$ of $\mathscr{C}$, $KO_*KO \underset{KO_*}{\otimes} M=\{KO_*KO \underset{KO_*}{\otimes} M^H\}$ is viewed as an object of $\mathscr{C}$ whose operation are represented by those of M as given in Lemma 5.1, where $\mathscr{C}=CRT$, CR, CT, RT, C, R or T. An object of $A\mathscr{C}$ is an object $M=\{M^H\}$ of $\mathscr{C}$ equipped with a KO_*KO-comodule structure map $\psi_M=\{\psi_M^H\}: M \to KO_*KO \underset{KO_*}{\otimes} M$ which commutes all the operations of M (cf. [9, 5.5]). A morphism of $A\mathscr{C}$ is a morphism $f=\{f^H\}$ of $\mathscr{C}$ compatible with the comodule structure maps. Whenever an object M is injective in $\mathscr{C}$, its extended comodule $KO_*KO \underset{KO_*}{\otimes} M$ is injective in $A\mathscr{C}$. Therefore the abelian category $A\mathscr{C}$ has enough injectives.

Fix a positive integer r such that it is congruent to ± 3 modulo 8 when $p=2$, and it generates the group of the units of Z/p^2 when p is an odd prime. For each prime p we can form the sequence of CW-spectra

$$S_{KZ_{(p)}} \xrightarrow{\iota_p} KOZ_{(p)} \xrightarrow{\psi_R^r-1} KOZ_{(p)} \xrightarrow{q} SQ \tag{5.1}$$

with trivial compositions (see [7, Theorem 4.3] or [9, 8.4]). Here $S_{KZ_{(p)}}$ is the $KOZ_{(p)*}$-localization of the sphere spectrum S, ψ_R^r is the stable Adams operation, the map ι_P is induced by the unit $\iota: S \to KO$ and the map q is associated with the inclusion $q_*: Z_{(p)} \subset Q$ in the homotopy group. Since the cofiber of the map ι_p coincides with the fiber of the map q, the above sequence (5.1) gives rise to the following fundamental exact sequence

(5.2)

$$0 \to \pi_*^{CRT} SZ_{(p)} \wedge W \to \pi_*^{CRT} KOZ_{(p)} \wedge W \to \pi_*^{CRT} KOZ_{(p)} \wedge W \to \pi_*^{CRT} SQ \wedge W \to 0$$

for any π_*^{CRT}-cofree KO-module spectrum W.

For each object M in $ACRT$ we consider the sequence

$$0 \longrightarrow M_{(p)} \xrightarrow{\psi_{M(p)}} KO_*KO \underset{KO_*}{\otimes} M_{(p)} \xrightarrow{(\psi_R^r \wedge 1-1)_* \otimes 1} KO_*KO \underset{KO_*}{\otimes} M_{(p)}$$

where $M_{(p)}=M \otimes Z_{(p)}$ and $\psi_{M_{(p)}}=\psi_M \otimes 1$ for the KO_*KO-comodule structure map ψ_M of M. Denote by $C(M_{(p)})$ the cokernel of the endomorphism $(\psi_R^r \wedge 1-1)_* \otimes 1$ on $KO_*KO \underset{KO_*}{\otimes} M_{(p)}$. From (5.2) it follows that

$$C(K_*^{CRT} X_{(p)}) \cong K_*^{CRT} X \otimes Q \quad \textit{if} \quad KO \wedge X \quad \textit{is} \quad \pi_*^{CRT}\textit{-cofree.} \tag{5.3}$$

Choose a short exact sequence $0 \to M \to K_*^{CRT} KO \wedge X \to N \to 0$ in $ACRT$ with $KO \wedge X$ π_*^{CRT}-cofree. Then an easy argument using the fundamental exact sequence (5.2) shows that the sequence

$$(5.4)\quad 0 \longrightarrow M_{(p)} \longrightarrow KO_*KO \underset{KO_*}{\otimes} M_{(p)} \longrightarrow KO_*KO \underset{KO_*}{\otimes} M_{(p)} \xrightarrow{q} C(M_{(p)}) \longrightarrow 0$$

is exact in $ACRT$. Moreover (5.3) implies that $C(M_{(p)})$ is a Q-module because $KO \wedge KO \wedge X$ is π_*^{CRT}-cofree by virtue of Lemma 4.2 i). Therefore the composite morphism

$$(5.5)\qquad M \otimes Q \xrightarrow{\psi_{M\otimes Q}} KO_*KO \underset{KO_*}{\otimes} (M \otimes Q) \xrightarrow{q} C(M_{(p)})$$

becomes an isomorphism in $ACRT$. Thus we obtain a fundamental exact sequence

$$(5.6)\quad 0 \longrightarrow M_{(p)} \longrightarrow KO_*KO \underset{KO_*}{\otimes} M_{(p)} \longrightarrow KO_*KO \underset{KO_*}{\otimes} M_{(p)} \longrightarrow M \otimes Q \longrightarrow 0.$$

By means of Theorem 1.4 and (5.5) we observe that

(5.7) *$M \otimes Q$ is injective in $ACRT$ if it is CRT-acyclic.*

Using the fundamental exact sequence (5.6) for each prime p and (5.7) as in the proof of [9, Theorem 7.3] we can easily show

THEOREM 5.2. *For each object M in $ACRT$, its injective dimension is at most 2 whenever M is CRT-acyclic.*

5.3. Owing to [12, Proposition 7] (or [17, Proposition 1.1] we have

PROPOSITION 5.3. *A CW-spectrum W is $K_*^{\mathscr{C}}$-injective if and only if the canonical morphism $\kappa_K^{\mathscr{C}}: [X, W] \to \mathrm{Hom}_{A\mathscr{C}}(K_*^{\mathscr{C}}X, K_*^{\mathscr{C}}W)$ is a monomorphism for any CW-spectrum X.*

We here give a few results concerning $K_*^{\mathscr{C}}$-injective spectra, which correspond to [17, Propositions 2.3, 2.4 and 2.5].

PROPOSITION 5.4. *If a CW-spectrum W is $K_*^{\mathscr{C}}$-injective, then it is a quasi KO-module spectrum such that $K_*^{\mathscr{C}}W$ is injective in $A\mathscr{C}$ and $\kappa_K^{\mathscr{C}}: [X, W] \to \mathrm{Hom}_{A\mathscr{C}}(K_*^{\mathscr{C}}X, K_*^{\mathscr{C}}W)$ is an isomorphism for any CW-spectrum X.*

PROOF. By virtue of Lemma 4.1 it is sufficient to show our result for any $\pi_*^{\mathscr{C}}$-cofree Z. Let Z be a KO-module spectrum which is $\pi_*^{\mathscr{C}}$-cofree. Then the homotopy $\pi_*^{\mathscr{C}}Z$ is cofree in $\mathscr{C}$ and hence it is injective in $\mathscr{C}$. Since the K-homology $K_*^{\mathscr{C}}Z$ is isomorphic in $A\mathscr{C}$ to the extended comodule $KO_*KO \underset{KO_*}{\otimes} \pi_*^{\mathscr{C}}Z$, it is certainly injective in $A\mathscr{C}$. By the $\mathscr{C}$-version of Lemma 1.5 we observe that $\kappa^{\mathscr{C}}: [X, Z] \to \mathrm{Hom}_{\mathscr{C}}(K_*^{\mathscr{C}}X, K_*^{\mathscr{C}}Z)$ is an isomorphism for any CW-spectrum X. Then it is immediate that $\kappa_K^{\mathscr{C}}: [X, Z] \to \mathrm{Hom}_{A\mathscr{C}}(K_*^{\mathscr{C}}X, K_*^{\mathscr{C}}Z)$ is an isomorphism for any CW-spectrum X.

Making use of Proposition 5.4 we can prove the following results by quite similar arguments to [17, Propositions 2.4 and 2.5].

Proposition 5.5. *If W is a KO_*-local spectrum such that $K_*^{\mathscr{C}}W$ is injective in $A\mathscr{C}$, then it is a $K_*^{\mathscr{C}}$-injective spectrum and* $\kappa_K^{\mathscr{C}}: [X, W] \to \mathrm{Hom}_{A\mathscr{C}}(K_*^{\mathscr{C}}X, K_*^{\mathscr{C}}W)$ *is an isomorphism for any CW-spectrum X* (see [9, Lemma 9.3]).

Proposition 5.6. *For each injective object I* of *$A\mathscr{C}$, there exists a $K_*^{\mathscr{C}}$-injective spectrum W_I whose K-homology $K_*^{\mathscr{C}}W_I$ is isomorphic in $A\mathscr{C}$ to the injective object I* (see [9, Lemma 9.2]).

Putting the above results together we obtain the following characterizations of $K_*^{\mathscr{C}}$-injective spectra (cf. [9, §9]).

Theorem 5.7. *For a KO-module spectrum W the following three conditions are equivalent:*
i) *W is a $K_*^{\mathscr{C}}$-injective spectrum,*
ii) *W is a $\pi_*^{\mathscr{C}}$-cofree spectrum, and*
iii) *$\pi_*^{\mathscr{C}}W$ is injective in $\mathscr{C}$.*

Proof. The implication i) $\to$ ii) follows from Proposition 4.6, and the implication ii) $\to$ iii) is immediate. On the other hand, the implication iii) $\to$ i) is shown by use of Proposition 5.5 because $K_*^{\mathscr{C}}W$ is injective in $A\mathscr{C}$ when $\pi_*^{\mathscr{C}}W$ is injective in $\mathscr{C}$. To show the final implication we may instead use Proposition 4.6 combined with Theorems 1.4, 3.2 and 3.4 by the aid of the $\mathscr{C}$-version of Lemma 1.5.

Theorem 5.8. *For a CW-spectrum W the following six conditions are all equivalent:*
i) *W is a $K_*^{\mathscr{C}}$-injective spectrum,*
ii) *W is a quasi KO-module spectrum such that $K_*^{\mathscr{C}}W$ is injective in $\mathscr{C}$,*
iii) *W is a quasi KO-module spectrum such that $K_*^{\mathscr{C}}W$ is injective in $A\mathscr{C}$,*
iv) *W is a K_*-local spectrum such that $K_*^{\mathscr{C}}W$ is injective in $A\mathscr{C}$,*
v) *$\kappa_K^{\mathscr{C}}: [X, W] \to \mathrm{Hom}_{A\mathscr{C}}(K_*^{\mathscr{C}}X, K_*^{\mathscr{C}}W)$ is an isomorphism for any CW-spectrum X, and*
vi) *$\kappa_K^{\mathscr{C}}: [X, W] \to \mathrm{Hom}_{A\mathscr{C}}(K_*^{\mathscr{C}}X, K_*^{\mathscr{C}}W)$ is a monomorphism for any CW-spectrum X.*

Proof. The implications i) $\to$ iii) $\to$ iv) $\to$ i) and i) $\to$ v) $\to$ vi) $\to$ i) follow immediately from Propositions 5.3, 5.4 and 5.5. On the other hand, Theorem 4.7 combined with Theorem 5.7 shows the equivalence between i) and ii).

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